Single and Double Black-Cox for pricing risky debt and equity with reorganization

Isabel Abínzano † Marcos Escobar‡ Luis Seco §

This draft: October 2007

Abstract

In this paper we add a Black and Cox (1976) approach for debt and equity valuation to the common practice of reorganization leading to a more realistic setting. We introduce first passage models in order to allow for both reorganization as well as liquidation before maturity time. This paper considers two types of settings. The first one is a first-passage approach for reorganization together with a Merton approach for default, while the second setting uses first-passage models for both reorganization and default. A comparison among the models proposed with Merton (1974) and Black and Cox (1976) approaches is provided.

Journal of Economic Literature classification: G12, G32, G33.

Keywords: First passage model, reorganization, liquidation.

†The first author gratefully acknowledges financial support from the Spanish Ministry of Education and Science (MEC, SEJ2006-14809-C03-01) and from a grant from Fundación ICO, and thanks the University of Toronto, where part of this work was done, for its hospitality. The usual caveat applies.

‡Assistant Professor at the Department of Financial Economics and Accounting, Universidad Pública de Navarra, Spain, isabel.abinzano@unavarra.es.

§Professor at the Department of Mathematics, University of Toronto, Canada, seco@math.toronto.edu.
1 Introduction

Reorganization is one of the two routes that a corporation in bankruptcy may take. When a corporation becomes insolvent and bankruptcy proceedings are commenced, the corporation is either liquidated or reorganized. In liquidation, the assets of the corporation are sold, either piecemeal or as a going concern. The proceeds from this sale are then divided among those who have rights against the corporation, with the division made according to the ranking of these rights. On the other hand, reorganization consists of a series of agreements between the company and its creditors which allow for the company to repay its debts and alter its structure to prevent the same problems from arising again. Therefore it can be seen as an attempt to extend the life of the company facing bankruptcy through special arrangements and restructuring.

For most companies, bankruptcy practices are governed by Chapter 11 of the U.S. Bankruptcy Code. The main purpose of Chapter 11 is to preserve the company as an operating concern while a plan for reorganization is worked out among creditors. Under a Chapter 11 filing the debtor is typically allowed to act as the trustee, in which case he retains complete control over the operation of the firm. This way, the debtor is entitled to file a reorganization plan during the first 120 days following the filing of a bankruptcy petition and has an additional 60 days to obtain acceptance by the creditors. When a firm is in financial distress, reorganization is often desirable because it avoids losses from liquidation or suboptimal incentive alignment of the existing contracts, as Bergman and Callen (1991) show.

Thus, the price of debt and equity of a firm should be affected by the possibility of reorganization. In the literature there are many articles studying the pricing of debt and equity of a firm. These studies fit within two broad approaches: Structural models and Reduced form models. In the so-called structural models there is a model definition of default, while in the reduced form models, the default is not causally modelled in terms of the firm’s assets and liabilities, but is typically given exogenously.

The best known model in the Structural framework is the traditional Merton (1974) contingent-claims-based approach for valuing corporate debt, which has become an integral part of the theory of Corporate Finance. In this approach, the equity value is
considered a call option on the value of the assets of the firm with strike price equal to the firm’s debt. However, this approach does not consider the possibility of reorganization when default occurs. Franks and Torous (1989) take into account the possibility of reorganization and provide a framework where the value of equity at maturity can be seen as an American call option on the firm’s assets value. Longstaff (1990) points out that many types of corporate reorganizations —such as Chapter 11 bankruptcy— can be viewed as the exercise of an extension privilege and suggests the application of the extendible option analysis to the pricing of the capital structure of the firm. Furthermore, Longstaff (1990) applies the flexible writer-extendible options to the pricing of the equity of a risky levered firm where bondholders have an incentive to extend the maturity date of the debt, while Abínzano and Navas (2007) apply the holder-extendible options defined by Longstaff (1990) to the pricing of the equity position of firms where shareholders have the possibility of filing a plan of debt restructuring.

Notwithstanding, these papers only permit default and/or reorganization at maturity, that is, the value of the firm is allowed to dwindle to nearly nothing without triggering a default. Black and Cox (1976) propose a more realistic model by allowing premature default. In their model, the default occurs as soon as the value of the firm’s assets reaches a lower threshold. In other words, the default time is defined by the first passage time of the firm value to some barrier. This first-passage-time model fits within the family of Structural models.

Examples of first-passage-time models are Longstaff and Schwartz (1995) and Cathcart and El-Jahel (1998), who extend the Black-Cox model by considering stochastic interest rates, the model of Zhou (2001) for calculating default correlations, and Fujita and Ishizaka (2002), where the first passage time of the firm value to some barrier is only considered to be the “caution time”, and the default occurs if some conditions are satisfied after the caution time.

Other first-passage-time models account for the possibility of a reorganization. This way, Francois and Morellec (2004) develop a model for pricing equity and debt when the firm can either liquidate their assets or renegotiate outstanding debt. The decision for liquidating or reorganizing depends only on the length of the period of time after crossing the default threshold. In Fan and Sundaresan (2000), reorganization implies
that debt holders swap their debt for equity. That means that the firm becomes an all-equity firm.

A more realistic setting would be that as soon as the firm assets value falls to some prescribed lower threshold, the firm could restructure its debt, for example forcing Chapter 11, changing its maturity and/or its face value. With this goal in mind, in this paper we develop a model for pricing debt and equity when the firm has the possibility of reorganization inspired on Black-Cox (1976), that is, restructuring or defaulting is allowed to occur before maturity as soon as some pre-agreed thresholds are exceeded. This way, we allow for an extension of the debt maturity as well as for a modification of the amount due. In other words, we extend Black and Cox (1976) in the same way Longstaff (1990) extends Merton (1974) to allow for reorganization.

The rest of the paper is organized as follows. Section 2 provides the main mathematical results. In Section 3 a single Black-Cox (1976) approach for reorganization is defined and studied. In the same section, an extension to allow for first-passage model for both reorganization and default is studied. A comparison of these models is provided in Section 4. Finally, Section 5 concludes.

2 Mathematical setting

As we have mentioned, in this paper we want to price the debt and equity of a firm with the possibility of reorganization as soon as the assets firm value falls to some lower threshold. Before the valuation, we need some mathematical setting that we present next.

The mathematical results needed are related to the joint distribution of the endpoint and the minimum of a Brownian motion with drift. The importance of this joint lies in the equivalence, within Structural models, between the time of default of a company and the minimum of the value of its assets. Therefore, several distributions related to the minimum of a lognormal process with constant drift and volatility are provided in the next two propositions.

Consider the following notation:

1. $V_t$ stands for the value of the assets of a company, while $X_t = \log(V_t)$. 
2. $V_R$ denotes the reorganization threshold, with $X_R = \log(V_R)$.

3. $V_L$ denotes the liquidation threshold, with $X_L = \log(V_L)$, $V_R > V_L$.

4. $K_1$ stands for the face value of debt, with $V_R \leq K_1$.

5. $K_2$ is the “re-assessed” debt face value, after debt has been reorganized, with $V_L \leq K_2$. Moreover, we assume that $K_2$ can be larger or lesser than $K_1$.

In deriving the next mathematical results, we assume the following conditions:

1. There is no recovery in the presence of default.

2. The interest rate, $r$, is known and constant through time.

3. The standard deviation of the return on the assets value, $\sigma$, is constant.

This way, we reach the next proposition, in which the main known results needed for this paper are outlined.

**Proposition 1.** Let $X_t$ be a stochastic process defined as follows:

\[ X_t = \mu t + \sigma W_t, \quad t > c, \quad X_c = X_0 \]  
(1)

\[ M_{n_t} = \min_{c \leq s \leq t} X_s \]  
(2)

\[ \tau_1 = \inf\{t > c : X_t = X_R\} \]  
(3)

\[ X_R < X_0 \]  
(4)

Where $W_t$ stands for an standard Brownian motion and $c$ is the initial date. Based on He et al. (1998), we obtain the following results:

1. The cumulative of $\tau_1$ is:

\[
P(\tau_1 > t) = P(M_{n_t} > X_R) = P(X_t > X_R, M_{n_t} > X_R) =
\]

\[
= P(X_t > X_R) - P(X_t > X_R, M_{n_t} < X_R)
\]

\[
= \Phi \left( \frac{X_0 - X_R + \mu (t - c)}{\sigma \sqrt{(t - c)}} \right) - \exp \left( \frac{2\mu (X_R - X_0)}{\sigma^2} \right) \Phi \left( \frac{X_R - X_0 + \mu (t - c)}{\sigma \sqrt{(t - c)}} \right)
\]

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Normal.
2. Differentiating with respect to $t$ we obtain the density of $\tau_1$:

$$f_{\tau_1}(t) = \phi \left( X_0 - X_R + \mu (t - c) \over \sigma \sqrt{(t - c)} \right) \left( {X_0 - X_R \over 2\sigma (t - c)^{3/2}} - {\mu \over 2\sigma \sqrt{(t - c)}} \right)$$

$$- \exp \left( {2\mu (X_R - X_0) \over \sigma^2} \right) \phi \left( {X_R - X_0 + \mu (t - c) \over \sigma \sqrt{(t - c)}} \right) \left( {X_R - X_0 \over 2\sigma (t - c)^{3/2}} - {\mu \over 2\sigma \sqrt{(t - c)}} \right)$$

(6)

where $\phi(\cdot)$ is the density function of the standard Normal distribution.

3. The joint density/distribution function of $(X_t, \tau_1)$, $p_{X, \tau_1}(x, t)dx$, can be obtained using Fokker-Planck equations, $(x \geq X_R)$:

$$P(X_t \in dx, \tau_1 > t) = p_{X, \tau_1}(x, t)dx =$$

$$= \phi \left( {x - \mu (t - c) \over \sigma \sqrt{(t - c)}} \right) \left[ 1 - \exp \left( - { (4(X_R - X_0)^2 - 4(X_R - X_0)x) \over 2\sigma^2 (t - c)} \right) \right] dx$$

(7)

We must make some observations about the previous results:

**Remark 1.**

1. If $\mu > 0$, then $P(\tau_1 > \infty) = 1 - \exp \left( {2\mu (X_R - X_0) \over \sigma^2} \right)$, that is, there is a positive probability of never crossing the threshold $X_R$.

2. If $\mu < 0$, then $P(X_R \leq X_t < \infty, \tau_1 < \infty) \approx 0$, what means that there is a small probability of both events, crossing a threshold and the endpoint being greater than $X_R$, happening simultaneously. The reason for this is that $P(X_R \leq X_t < \infty) \approx 0$.

3. The assumption of constant volatility is required for the proposition to apply. On the other hand, the assumptions of zero recovery and constant interest rate could be relaxed, what would require to use some challenging, from the computational side, joint distributions, which could be found in He et al. (1998).
As it can be inferred from Proposition 1, the variable \( \tau_1 \) accounts for the time where the reorganization threshold is crossed. Another important variable is the time where the liquidation threshold is crossed, \( \tau_2 \). Define:

\[
\tau_2 = \inf \{ t > \tau_1 : X_t = X_L \}, \quad X_L < X_R
\]  

(8)

This way, we can enunciate the following result:

**Proposition 2.** The joint cumulative distribution for \((\tau_1, \tau_2)\) is:

\[
P(\tau_1 < t_1, \tau_2 < t_2) = \int_0^{t_1} P(\tau_2 < t_2 | \tau_1 = s_1) f_{\tau_1}(s_1) ds_1
\]  

(9)

\[
P(\tau_2 < t_2 | \tau_1 = s_1) = \Phi \left( \frac{X_R - X_L + \mu(t_2 - s_1)}{\sigma \sqrt{t_2 - s_1}} \right) - \exp \left( \frac{2\mu(X_L - X_R)}{\sigma^2} \right) \Phi \left( \frac{X_L - X_R + \mu(t_2 - s_1)}{\sigma \sqrt{t_2 - s_1}} \right)
\]  

(10)

where \( f_{\tau_1}(s_1) \) is the density function found in Proposition 1 while \( P(\tau_2 < t | \tau_1 = s_1) \) is the density function of a Brownian motion starting at \( X_{s_1} = X_R \) with threshold \( X_L \).

Note that differentiating equation (9) with respect to \( t_1 \) and \( t_2 \) would lead to the joint density of \((\tau_1, \tau_2), f_{\tau_1, \tau_2}(t_1, t_2)\).

3 Risky debt analysis

In what follows, we describe the approaches of Merton (1974) and Black and Cox (1976). Afterwards two generalizations of the latter, accounting for reorganization, are analyzed. We will start with debt pricing as well as equity pricing with no reorganization from the point of view of Merton (1974). Then the same description will be applied to the Black and Cox (1976) setting. In this context, it should be emphasized that our approach would be a generalization of Black and Cox (1976) to the idea of reorganization in the same spirit of what Longstaff (1990) does to the variant of Merton (1974).

In our setting, the amounts \( K_1 \) and \( K_2 \), as well as the distress thresholds, \( V_R \) and \( V_L \), are inputs to the problem. As noted by Leland (1994) and Ericson and Reneby (1998), there are several ways to determine and justify a distress threshold. An economic
approach views the distress threshold as the level of assets value necessary for the firm to retain sufficient credibility to continue operations. A contractual interpretation for the existence of the distress threshold is based on positive net worth covenants that enable bondholders to force reorganization or liquidation in the event the value of the firm falls below a pre-determined threshold. Thus, we can see $V_R$ and $V_L$ as wake-up calls to the company that the value of its assets are too low to keep operating in normal conditions. An example would be to take $V_R$ and $V_L$ as $K_1$ and $K_2$, respectively.

3.1 No reorganization: Merton (1974) and Black and Cox (1976)

Merton (1974) introduces the view that equity value is a call option on the value of the assets of the firm with strike price equal to the face value of the firm’s debt while a risky bond is equivalent to a long position in a risk-free bond plus a short put option. In this context, default of the company occurs only at maturity $T_1$.

Let us consider a firm financed with shares of stock and one zero-coupon bond with face value $K_1$ and maturity $T_1$. The payoff expected by shareholders at $T_1$ shall be:

$$(V_{T_1} - K_1) 1_{\{V_{T_1} > K_1\}}$$

while the payoff at maturity from the view of debtholders shall be:

$$K_1 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{V_{T_1} \leq K_1\}}$$

These payoffs are random variables. Thus, if the value of the company is larger than its debt, debtholders get $K_1$ while shareholders get $V_{T_1} - K_1$, and in case of default, the debtholders receive what is left of the company, $V_{T_1}$, while the shareholders get 0.

On the other hand, Black and Cox (1976) allows default to occur at anytime previous to maturity. This way it protects debtholders at the expense of equityholders. In this case, the payoff expected by shareholders at $\min(T_1, \tau_1)$ shall be:

$$(V_{T_1} - K_1) 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}}$$

while from the view of debtholders, the payoff at $\min(T_1, \tau_1)$ shall be:

$$K_1 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} \leq K_1\}} + V_R 1_{\{\tau_1 \leq T_1\}}$$
with \( V_L \leq K_1 \). In other words, as soon as the value of the company is lower than its debt, debtholders get \( K_1 \) while shareholders get 0. On the contrary, if no default occurs, shareholders get \( V_{T_1} - K_1 \) while debtholders keep getting the value of the debt \( K_1 \).

### 3.2 Single Black-Cox risky debt with reorganization

Before deriving our approach, we must make two main assumptions:

1. A reorganization is allowed as soon as the asset’s value crosses a threshold, \( V_R \), with \( \tau_1 \leq T_1 \). The maturity time as well as the strike price are changed at this stage to \( T_2 \) and \( K_2 \), respectively.

2. Default may occur only at \( T_2 \) if and only if \( V_{T_2} < K_2 \).

It is also assumed that \( V_R \leq K_1 \), from Section 2, which means that reorganization makes sense if and only if the assets of the company are not enough to cover the debt. We must also remark that the new face value of debt, \( K_2 \), could be less than the initial face value, \( K_1 \). This is consistent with Haugen and Senbet (1978, 1988), who argue that debtholders could offer to reduce the debt claim so as to avoid bankruptcy and its associated costs.

#### 3.2.1 Shareholders viewpoint

From the point of view of shareholders (SH), the payoff condition satisfied at \( \min(T_1, \tau_1) \) is:

\[
(V_{T_1} - K_1)1_{\{\tau_1 > T_1\}}1_{\{V_{T_1} > K_1\}} + c(V_R, K_2, T_2 - \tau_1)1_{\{\tau_1 \leq T_1\}}
\]  

where \( c(V_R, K_2, T_2 - \tau_1) \) denotes the price of a European call option.

**Proposition 3.** Under previous assumptions, the price of a single Black-Cox equity with reorganization is:

\[
SBC_{SH}(V_0, V_R, K_1, K_2, T_1, T_2) = e^{-rT_1}E_0 \left[ (\exp(X_{T_1}) - K_1)1_{\{\tau_1 > T_1\}}1_{\{X_{T_1} > \ln K_1\}} \right]
\]
\[ + E_0 \left[ e^{-r\tau_1} c(V_R, K_2, T_2 - \tau_1) 1_{\{\tau_1 < T_1\}} \right] = \\
= e^{-rT_1} \int_{\ln K_1}^{\infty} (\exp(x) - K_1) p_{X, \tau_1}(x, T_1) \, dx + \int_0^{T_1} e^{-rt} c(V_R, K_2, T_2 - t) f_{\tau_1}(t) \, dt \]  \tag{16}

where:
\[ c(V_R, K_2, T_2 - t) = V_R \Phi(a) - K_2 e^{-r(T_2 - t)} \Phi(b) \]  \tag{17}

with:
\[ a = \ln \frac{V_0}{K_2} + \left( r + \frac{1}{2} \sigma^2 \right) (T_2 - t) \frac{1}{\sigma \sqrt{T_2 - t}} \]  \tag{18}
\[ b = a - \sigma \sqrt{T_2 - t} \]  \tag{19}

and with \( f_{\tau_1}(t) \) and \( p_{X, \tau_1}(x, T_1) \) defined by (6) and (7).

**Proof.** This follows by substituting properly from propositions 1 and 2. \qed

**Remark 2.** The "price of reorganizing" for shareholders, within Black-Cox framework, would be the difference between the value of SBC and BC:
\[ E_0 \left[ e^{-r\tau_1} c(V_R, K_2, T_2 - \tau_1) 1_{\{\tau_1 < T_1\}} \right] \]  \tag{20}

### 3.2.2 Debtholders viewpoint

On the other hand, from the point of view of debtholders (DB), the payoff condition satisfied at \( \min(T_1, \tau_1) \) is:
\[ K_1 1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} > \ln(K_1)\}} + V_{T_1} 1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} \leq \ln(K_1)\}} + \\
( K_2 e^{-r(T_2 - \tau_1)} - p(V_R, K_2, T_2 - \tau_1) ) 1_{\{\tau_1 \leq T_1\}} \]  \tag{21}

where \( p(V_R, K_2, T_2 - \tau_1) \) denotes the price of a European put option.

**Proposition 4.** Under previous assumptions the price of a single Black-Cox risky debt with reorganization is:
\[ SBC_{DB}(V_0, V_R, K_1, K_2, T_1, T_2) = \\
= e^{-rT_1} K_1 E_0 \left[ 1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} > \ln(K_1)\}} \right] + e^{-rT_1} E_0 \left[ \exp(X_{T_1}) 1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} \leq \ln(K_1)\}} \right] \]
\[ + K_2 E_0 \left[ e^{-rT_2} 1_{\{T_1 < T_2\}} \right] - E_0 \left[ e^{-r\tau_1} p(V_R, K_2, T_2 - \tau_1) 1_{\{\tau_1 < T_1\}} \right] \\
= e^{-rT_1} K_1 \int_{\ln K_1}^{\infty} p_{X,T_1}(x, T_1) dx + e^{-rT_1} \int_{-\infty}^{\ln K_1} \exp(x) p_{X,T_1}(x, T_1) dx \\
+ K_2 e^{-rT_2} \int_0^{T_1} f_{\tau_1}(t) dt - \int_0^{T_1} e^{-r t} p(V_R, K_2, T_2 - t) f_{\tau_1}(t) dt \] (22)

where:

\[ p(V_R, K_2, T_2 - t) = K_2 e^{-r(T_2 - t)} \Phi(-b) - V_R \Phi(-a) \] (23)

with:

\[ a = \ln \left( \frac{V_R}{K_2} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T_2 - t) \]
\[ b = a - \sigma \sqrt{T_2 - t} \] (24)

and with \( f_{\tau_1}(t) \) and \( p_{X,T_1}(x, T_1) \) defined by (6) and (7).

**Proof.** This follows by substituting properly from propositions 1 and 2. \( \square \)

**Remark 3.** The “price of reorganizing” for debtholders, within Black-Cox framework, would be the difference between the value of SBC and BC:

\[ E_0 \left[ e^{-r\tau_1} \left( K_2 e^{-r(T_2 - \tau_1)} - p(V_R, K_2, T_2 - \tau_1) - V_R \right) 1_{\{\tau_1 \leq T_1\}} \right] \] (26)

### 3.3 Double Black-Cox risky debt with reorganization

This approach consists of a generalization of the Single Black-Cox in the sense that after reorganization the default may occur at anytime before the newly set maturity time. As before, there would be two main assumptions in this case:

1. A reorganization is allowed as soon as the value of the assets crosses a threshold, \( V_R \), with \( \tau_1 \leq T_1 \). Both the maturity time and the strike price are changed at this stage to \( T_2 \) and \( K_2 \), respectively.

2. Default may occur either before maturity \( T_2 \) as soon as \( V_t < V_L \), with \( V_L < V_R \) and \( t > \tau_1 \), or it can occur at maturity if \( V_{T_2} < K_2 \).
3.3.1 Shareholders viewpoint

From the point of view of shareholders, the payoff condition satisfied at \( \min(T_1, \tau_1) \) is:

\[
(V_{T_1} - K_1)1_{\{\tau_1>T_1\}}1_{\{V_{T_1}>K_1\}} + cFPM(V_R, K_2, V_L, T_2 - \tau_1)1_{\{\tau_1<T_1\}}
\]  \( (27) \)

where \( cFPM(V_R, K_2, V_L, T) \) is a first passage model European option and \( V_R = V(0) \), that is:

\[
cFPM(V_R, K_2, V_L, T) = e^{-rT}E_0 \left[ \exp(X_{T_1}) - K_2 \right]1_{\{\tau_2>T\}}1_{\{X_{T_1}>\ln(K_2)\}}
\]

\[
= e^{-rT} \int_{\ln(K_2)}^{\infty} (\exp(x) - K_2) p_{X, \tau_2}(x, T) dx \tag{28}
\]

**Proposition 5.** Under previous assumptions, the price of a double Black-Cox equity with reorganization is:

\[
DBC_{SH}(V_0, V_R, V_L, K_1, K_2, T_1, T_2)
\]

\[
= e^{-rT_1}E_0 \left[ \exp(X_{T_1}) - K_1 \right]1_{\{\tau_1>T_1\}}1_{\{X_{T_1}>\ln(K_1)\}}
\]

\[
+ E_0 \left[ e^{-r\tau_1}cFPM(V_R, K_2, V_L, T_2 - \tau_1)1_{\{\tau_1<T_1\}} \right]
\]

\[
= e^{-rT_1} \int_{\ln(K_1)}^{\infty} (\exp(x) - K_1) p_{X, \tau_1}(x, T_1) dx
\]

\[
+ e^{-rT_2} \int_0^{T_1} \int_{\ln(K_2)}^{\infty} (\exp(x) - K_2) p_{X, \tau_2}(x, T_2 - t_1)f_{\tau_1}(t_1)dxdt_1 \tag{29}
\]

**Proof.** This follows by substituting properly from propositions 1 and 2. \( \square \)

**Remark 4.** The “price of reorganizing” for shareholders, within Black-Cox framework, would be the difference between the value of \( DBC \) and \( BC \):

\[
E_0 \left[ e^{-r\tau_1}cFPM(V_R, K_2, V_L, T_2 - \tau_1)1_{\{\tau_1<T_1\}} \right] \tag{30}
\]
3.3.2 Debtholders viewpoint

From the point of view of debtholders, the payoff condition at $\min(T_1, \tau_1)$ is:

$$K_1 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} \leq K_1\}} +$$

$$+ E_{\tau_1} \left[ e^{-r(\tau_2 - \tau_1)} K_2 1_{\{\tau_2 > T_2\}} 1_{\{V_{\tau_2} > K_2\}} + e^{-r(\tau_2 - \tau_1)} V_T 1_{\{\tau_2 > T_2\}} 1_{\{V_{\tau_2} \leq K_2\}} + e^{-rT_2} V_L 1_{\{\tau_2 \leq T_2\}} \right] \cdot 1_{\{\tau_1 \leq T_1\}} \quad (31)$$

**Proposition 6.** The price of a double Black-Cox risky debt with reorganization is:

$$DBC_{DH}(V_0, V_R, V_L, K_1, K_2, T_1, T_2) =$$

$$= e^{-rT_1} K_1 E_0 \left[ 1_{\{\tau_1 > T_1\}} 1_{\{X_{\tau_1} > \ln(K_1)\}} \right] + e^{-rT_1} E_0 \left[ \exp(X_{\tau_1}) 1_{\{\tau_1 > T_1\}} 1_{\{X_{\tau_1} < \ln(K_1)\}} \right]$$

$$+ K_2 e^{-rT_2} E_0 \left[ 1_{\{\tau_2 > T_2\}} 1_{\{X_{\tau_2} > \ln(K_2)\}} \right]$$

$$+ e^{-rT_2} E_0 \left[ \exp(X_{\tau_2}) 1_{\{\tau_1 < T_1\}} 1_{\{\tau_2 > T_2\}} 1_{\{X_{\tau_2} < \ln(K_2)\}} \right] + V_L E_0 \left[ e^{-rT_2} 1_{\{\tau_1 < T_1\}} 1_{\{\tau_2 < T_2\}} \right]$$

$$= e^{-rT_1} K_1 \int_{\ln K_1}^{\infty} p_{X, \tau_1}(x, T_1) dx + e^{-rT_1} \int_{-\infty}^{\ln K_1} \exp(x) p_{X, \tau_1}(x, T_1) dx$$

$$+ e^{-rT_2} K_2 \int_{0}^{T_1} \left( \int_{\ln K_2}^{\infty} p_{X, \tau_2}(x, T_2 - t_1) dx \right) f_{\tau_1}(t_1) dt_1$$

$$+ e^{-rT_2} \int_{0}^{T_1} \left( \int_{-\infty}^{\ln K_2} \exp(x) p_{X, \tau_2}(x, T_2 - t_1) dx \right) f_{\tau_1}(t_1) dt_1$$

$$+ V_L \int_{0}^{T_1} \left( \int_{0}^{T_2 - t_1} e^{-rT_2} f_{\tau_2}(t_2) dt_2 \right) f_{\tau_1}(t_1) dt_1 \quad (32)$$

**Proof.** This follows by substituting properly from propositions 1 and 2. \(\Box\)

**Remark 5.** The "price of reorganizing" for shareholders, within Black-Cox framework, would be the difference between the value of DBC and BC:

$$E_0 \left[ \left( E_{\tau_1} \left[ e^{-r(\tau_2 - \tau_1)} K_2 1_{\{\tau_2 > T_2\}} 1_{\{X_{\tau_2} > \ln(K_2)\}} + e^{-r(\tau_2 - \tau_1)} \exp(X_{\tau_2}) 1_{\{\tau_2 > T_2\}} 1_{\{X_{\tau_2} < \ln(K_2)\}} + e^{-rT_2} V_L 1_{\{\tau_2 \leq T_2\}} \right] \right) - V_R \right) 1_{\{\tau_1 \leq T_1\}} \right] \quad (33)$$
4 Empirical results

Once the expressions for pricing debt and equity in the event of a reorganization have been obtained, we provide several empirical results with the aim of clarifying the impact that variables, parameters and the various approaches have on the credit spreads.

First, in Table 1 and Table 2 we provide two examples of pricing debt and equity using the four approaches we have studied: Merton (1974) (expressions 11 and 12), Black and Cox (1976) (expressions 13 and 14) and the Single (expressions 16-19 and 22-25) and Double Black-Cox models (expressions 29 and 32) proposed in this paper. In Table 1, the value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The company value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ (70\% of company’s value) which equals the debt amount at first maturity, that is $K_1 = 0.7$, and $V_L = 0.5$, respectively. The debt amount at second maturity in case of reorganization is $K_2 = 0$, which equals $K_1$. In Table 2, $K_2$ is smaller than $K_1$, that is, debtholders offer a reduction in the face value of debt. It can be noticed that in both tables the value obtained for shareholders with Black and Cox (1976) is lower than the value given by Merton (1974). This result is reasonable because there is a higher probability of default in the former structure compare to Merton’s approach. Moreover the value obtained by using the Double Black-Cox is lower than the one obtained by using Single Black-Cox. The reasoning for this result is similar, because allowing for defaults before maturity increases the likelihood of defaults and therefore puts more pressure on shareholders. For bondholders the opposite is obtained.

Next, we describe the impact that some meaningful parameters, like maturity time, the face value of debt and the thresholds, have on the credit spread of the firm. Figure 1 describes credit spreads versus $T_1$ for all four approaches. The assumptions are as in Table 1 but with $T_2 = 2T_1$ (this selection is based in common practices and it is used only for easiness of explanation). It is observed that as maturity increases there exists a clear difference between all four approaches, where Single Black-Cox approach gives the biggest credit spread followed by Merton, Double BC and BC. Figure 2 plots the credit spreads versus the ratio $K_2/K_1$ for all four approaches. The assumptions are as in
Table 1. This ratio gives an idea of which method could be more beneficial for debtholders. The Single Black-Cox approach is more risky when the face value of debt decreases after reorganizing, that is, it gives a higher credit spread, while it is as risky as the Double Black-Cox approach when the face value of debt increases after reorganization. In Figure 3 we describe credit spreads versus the distance between maturity times, $T_2 - T_1$, for all four approaches. The assumptions are as in Table 1 but with $T_1 = 1$. The result is similar to those in Figure 1 but more focused on the maturity time after reorganization. Once again Single BC is the more risky approach, leading to important differences as the second maturity time increases. This implies that short waiting times between reorganization and final maturity is optimal for SBC approach.

Finally, in Figures 4 and 5, the impact of the thresholds, $V_R$ and $V_L$, on the credit spread are shown. The assumptions are as in Table 1. In Figure 4 we plot credit spreads versus $V_R$ for all four approaches. We must remark that $V_R \in (V_L, V_0)$. It can be seen that the Single Black-Cox approach is still the most expensive in these situations, that is, it has the biggest credit spread. Figure 5 describes credit spreads versus $V_L$ for all four approaches. We should notice that $V_L \in (0, V_R)$ and, as expected, three methods are constant, Merton (1974), Black and Cox (1976) and the Single Black-Cox model, that is, there is no influence of $V_L$.

5 Conclusions

In this paper we have developed two first-passage-time models for pricing risky debt and equity when the firm has the possibility of carrying out a reorganization. These two approaches generalize the Black and Cox (1976) approach, in which default is the only possibility for a firm in case of financial distress. In practice, reorganization consists of an alternative to liquidation as an attempt to preserve the company as an operating concern.

The first model we propose is the Single Black-Cox approach. Within this approach, a reorganization of the firm is started as soon as the value of the firm crosses a lower threshold. We allow for a change in the face value of debt as well as in the time to maturity. This way, we do not impose any restriction on the new face value of debt, that
is, it can be bigger, equal or lower than the initial value. This is consistent with Haugen and Senbet (1978, 1988), who argue that debtholders can offer to reduce the debt claim in order to avoid bankruptcy. The Single Black-Cox analysis is more realistic than other first-passage models because it allows for reorganization before the initial maturity, and it does not restrict the characteristics of the new debt.

Second, we propose the Double Black-Cox approach, that consists of a generalization of the Single Black-Cox approach in the sense that after reorganization the default may occur at anytime before the new maturity time. In this model, we also allow for changes in the face value of debt and in time to maturity. This seems to be a more realistic approach, because it allows default as soon as the value of the assets of the firm is too low, that is, the firm can default without waiting until the new maturity time.

Finally, once we have proposed the expressions for pricing debt and equity when there exists the possibility of a reorganization, we provide several empirical results with the aim of clarifying the impact that variables and approaches have on the credit spread for debtholders and shareholders. We price debt and equity by using Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed here. We show that two of these approaches, Black and Cox (1976) and Double Black-Cox, are beneficial for bondholders in the sense that give protection against large losses, while they are detrimental for shareholders. On the other hand, we observe that Merton (1974) and Single Black-Cox settings could be more risky to bondholders because of the probability of the values of the company to be very small.
References


Table 1: Pricing of debt and equity using four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The asset value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$. 

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<tr>
<td>Merton($V_0, V_R, K_1, T_1$)</td>
<td>0.3574</td>
<td>0.6426</td>
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<tr>
<td>Black-Cox($V_0, V_R, K_1, T_1$)</td>
<td>0.3000</td>
<td>0.7000</td>
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<td>0.5760</td>
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Table 2: Pricing of debt and equity using four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is \(\sigma = 0.5\) and the risk-free interest rate is \(r = 0\). The assets value at zero is \(V_0 = 1\). The first maturity date is \(T_1 = 1\) and the second maturity date, in case of reorganization, is \(T_2 = 2\). The thresholds for reorganization and for liquidation are \(V_R = 0.7\) and \(V_L = 0.5\), respectively. The debt amount at first maturity is \(K_1 = 0.7\) and at second maturity is \(K_2 = 0.5\).
Figure 1: Effect of Maturity time on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The second maturity date is $T_2 = 2T_1$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$. 
Figure 2: Influence of the ratio $K_2/K_1$ on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively.
Figure 3: Influence of $T_2 - T_1$ on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$. 
Figure 4: Influence of $V_R$ on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The threshold for liquidation is $V_L = 0.5$. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$. 
Figure 5: Influence of $V_L$ on the Credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The threshold for reorganization is $V_R = 0.7$. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$. 