Investment and Credit Risk: a Structural Approach

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ABSTRACT

The paper investigates the impact on credit risk of capital structure choices driven by firm’s investments and financing decisions. We propose a realistic dynamic structural model featuring endogenous investment, capital structure and default. We calibrate the model on accounting and market data. Using simulation, we find that, credit spreads as well as other standard metrics of credit worthiness, like quasi-market leverage, default rate, and the distribution of firms across rating classes, are well fitted by the model. We find that the introduction of investment flexibility has the largest impact on credit risk among the studied features, because of an inherent under-investment agency cost created by debt and because equityholders do not follow a value maximizing capital structure policy.

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Introduction

Since Merton (1974), the theoretical framework most widely used to analyze the determinants of credit risk on corporate debt has been the contingent claim model because it can incorporate: the effect of corporate as well as personal taxation, the effect of different debt covenants, dynamic capital structure with different types of transaction costs and default decisions, including strategic debt service, with different forms of recovery functions. Moreover, the model can be extended to stochastic interest rates and generally more sophisticated pricing environments than the original Black and Scholes model. Actually, the goal of theoretical literature on capital structure and credit risk has been to extend the original contingent claim model along the above mentioned (and many other) directions.

Empirical researchers have implemented some of these models to analyze to what extent their predicted patterns and figures for credit risk fit the data. Overall, empirical studies find that the analyzed structural models tend to generate too low yield spreads on low leverage firms with low business risk, and too high yield spreads for firms in high risk classes. Moreover, structural models with endogenous capital structure decisions tend to predict higher leverage levels than are empirically observed. The intuition is that the structural models that have been considered so far miss some important components of the debt valuation problem.

Traditionally, structural models of capital structure follow the Modigliani and Miller (1958) and (1963) assumption that investment is exogenous, independent of financing decisions. However, investment does influence financing decisions, as the gap between capital expenditure and internal funds is a major determinant of debt issuance decisions and because cash flows generated by current investments will affect the future need of funds.

On the other hand, existing debt influences the optimal investment policy. In a static framework, the conventional wisdom is that equityholders can extract value from debtholders by increasing assets risk and exploiting limited liability (asset substitution), as explained by Jensen and Meckling (1976). Alternatively, equityholders may forego profitable investment opportunities because, while they pay the investment cost, the value increase would also accrue to existing debtholders and only in part to themselves (debt overhang), as argued by Myers (1977).

The result is that the capital structure decision is based on the trade-off between the benefits and the costs of debt financing, including the agency cost due to sub-optimality.

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1These unsatisfactory results of structural models can partly be explained by the non-default components in corporate bond yield spreads, like the liquidity premium, as proposed by Longstaff, Mithal, and Neis (2005), or taxes and systematic risk, as showed by Elton, Gruber, Agrawal, and Mann (2001). Since we are interested in a thorough description of credit risk, the effect of credit risk on yield spread is only one of the components of the effect we observe.
of an equity maximizing investment policy with respect to a value maximizing policy, which would be also in the interest of debtholders.

The case of a “static” capital structure decision with agency costs can be described as follows: if a currently unlevered firm issues debt, the agency cost given by (current and future) sub-optimal investments would be rationally incorporated into debt price. Equityholders account for that cost in their optimization and adapt their overall investment and financing policy accordingly. Note that in a model with “static” capital structure, the firm always decides (also in a recursive multiperiod setting) the debt policy when it is currently unlevered. This case has been studied by many authors.2

Unlike in “static” capital structure models, in a “dynamic” setting a firm may have existing debt which affects both the investment decision (either in the sense of over-investment or under-investment) and the capital structure decision. Also, the decision to change the current debt level is made to maximize equity value. For this reason, existing debt may create an incentive for equityholder to deviate from a first best capital structure policy.

To the best of our knowledge, there are only two papers that deal with “dynamic” capital structure and dynamic investment. Sundaresan and Wang (2006) present a simplified model where, in a real options setting, a firm with two compound (investment plus expansion) options can issue a blend of debt and equity to finance the capital expenditure at the exercise dates. Debt is a non-callable perpetuity and the issuance is contingent on the investment and on the expansion decision. Debt issued first is more senior than debt issued later on to finance the expansion. At the time of the first issuance, the firm is unlevered. So the capital structure decision is “static” in the sense defined above and it maximizes equity value, which in this case is equal to the total firm value. When the second portion of debt is issued, the capital structure decision together with the investment decision are made to maximize the equity value, which is different from the firm value because of the presence of more senior debt. Both the capital structure

2Leland (1998) presents a simplified model of risk shifting behavior of equityholders. After issuing the debt, the shareholders can decide the risk of the asset either to maximize the equity value or the firm value. The “static” capital structure is chosen at the initial date to maximize total firm value. In Childs, Mauer, and Ott (2005), the investment side of the model is given by a growth option to alternatively expand or replace the asset in place. The option can be exercised to maximize either the equityholders or total firm value. At the initial date, the debt policy (maturity of debt) is decided under a first best perspective. At subsequent dates, the firm is allowed to change the debt level only at fixed maturities. At these dates, the firm is unlevered, and the agency cost incorporated into debt price is only the one related to sub-optimality of the investment policy. In this sense the setting is “static” as defined in the text. Moyen (2007) studies the impact of underinvestment in a framework very similar to ours. The firms dynamically adapts the production capacity (investment/disinvestment) and changes the debt level. The goal of her model is to compare the value given by a first best investment policy to the value of a second best investment policy. Since she considers one period zero-coupon debt and the capital structure decision aims at maximizing the firm value at each t, what we said for Childs, Mauer, and Ott (2005) still applies.
and the investment decisions may be sub-optimal from senior debtholders’ perspective. Hence, the price of debt at the time of the first investment/financing decision incorporates both components of agency cost. Also Titman and Tsyplakov (2005) present a model with dynamic investment and “dynamic” capital structure, although with the purpose of measuring the agency cost due to conflict of interest between equityholders and bondholders on the investment policy. Importantly, their second best model is based on dynamic investment and financing decisions made to maximize equityholders value. This is what we implement also in our model.

We want to incorporate dynamic investment decisions in a realistic setting with “dynamic” capital structure, where not only investment but also financing is dynamically decided to maximize equityholders value. In this setting, necessarily based on long term callable debt, agency costs of debt would reflect sub-optimality of both investment and financing decisions, with respect to optimality of firm value maximizing decisions. Differently from Titman and Tsyplakov (2005) and from Sundaresan and Wang (2006), we focus on the effect that this costs have on the credit risk measures.\(^3\)

This work extends the existing literature by exploring the joint effects of dynamic capital and investment decisions on credit risk within an infinite horizon discrete-time stochastic framework. Investment can be financed using operating cash flows or external (both debt and/or equity) funds. Hence, there is also dynamic choice of capital structure and endogenous default. Likewise, the effects of corporate and personal taxation, costly financial distress, debt/equity issuance, and bankruptcy costs are included.

Although our work is related to a vast literature on capital structure and credit risk, it is particularly indebted to Hennessy and Whited (2007) and Moyen (2007). Their model describes the interaction between a firm’s investment and financing policies with endogenous default risk within a one-period debt model with endogenous investment. Actually, using one-period debt results in a significant simplification because the debt can be changed without incurring debt adjustment costs and agency costs due to sub-optimality of the financing policy. Our model differs from theirs because we use infinite-maturity debt and consider also the effect of investment irreversibility.

Given the simultaneous investment and financing decisions, the solution of the valuation problem for equity and debt is the fixed-point of a two-dimensional Bellman operator. Since our model includes perpetual (as opposed to one-period) debt, we cannot rely on the solution approach introduced by Cooley and Quadrini (2001) and extended by Moyen (2007) and Hennessy and Whited (2007) to solve the problem. Therefore, we introduce an efficient numerical method, which as far as we know is new to this field.

To calibrate the model, we compute the empirical credit risk metrics using firm accounting information (Compustat North America Industrial Annual), share prices (Cen-\(^3\)In Section I.E we will provide more details about the differences between our model and the ones by Titman and Tsyplakov (2005) and Sundaresan and Wang (2006).
ter for Research on Security Prices - CRSP), default rates (Moody’s Investors Service (2006)), and credit spread paid by industrial bonds (Reuters).

In contrast to the previous literature, we calibrate the model by fitting the empirical credit spreads for different risk classes. We think this is superior to matching the whole sample average credit spread or the investment bonds and speculative bonds average credit spreads because we can better capture the heterogeneity of firms belonging to different credit classes. The calibration procedure is based on a simulated sample of companies sorted into seven whole letter risk classes (from AAA to C). A simulated company is placed in a class based on its quasi-market leverage. Then we compute the average credit spread for each risk class, and finally, the distribution of credit spreads is produced. After the best-fit parameters are obtained, we analyze how different measures of credit risk (credit spreads, default rates per-year, and the credit class distribution) for the simulated sample based on those parameters compare to the empirical ones.

Our results show that the agency costs related to investment and to capital structure policy are the main determinants of capital structure decisions and of the yield spreads. On the other hand, dynamic capital structure decisions made only to exploit the benefits related to interest tax deduction have only a second order effect on credit risk. Likewise, the credit spreads estimated using the model including endogenous investment and capital structure better approximate the empirical credit spreads. Moreover, the empirical distribution of firms to be found in the different risk classes produced by our model better fits the corresponding empirical distribution. This means that the model can correctly predict the empirical leverage and the empirical credit spreads simultaneously. Similarly, improved results are found for the default rates.

The outline of our work is as follows. In Section I, we present the model and some restricted versions of the model itself, in order to analyze the effect of additional control levers. Moreover, we described the agency issues inherent to a model with dynamic capital structure and investment in an equity maximizing setting. In Section II we analyze the determinants of credit risk, as reflected in the credit spread. Then, after describing the simulation procedure and the calibration using empirical data, we compare the distributional properties of the credit risk metrics generated by our model to the one generated by models restricted in the investment and financing policies. In Section III we offer our concluding remarks. Appendix A provides the details of the numerical procedure we use to solve the model.

I. The model

In this section, we introduce the valuation model for corporate securities in a realistic setting with endogenous investment, dynamic capital structure decision and default.
A. Economic environment

The source of uncertainty is the productivity of firm’s capital stock, denoted $\theta$. We assume that $\theta$, under the risk–neutral probability follows a logarithm AR(1) process of the form

$$\log \theta_{t+1} = \eta + \rho \log \theta_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \sigma), \quad |\rho| < 1. \quad (1)$$

The process in Equation (1) is motivated by the fact that there is empirical evidence that earnings are persistent.\(^4\)

As a consequence, the EBITDA (operating cash flow before taxes), denoted $\pi(k, \theta)$, which depends on both the book value of assets, $k > 0$, and the shock, $\theta$, is

$$\pi(k, \theta) = \theta k^\alpha - F, \quad (2)$$

where $F > 0$ is a fixed cost that summarizes all expenses, and $\pi$ exhibits decreasing returns to scale ($\alpha < 1$). We assume that the firm cannot change the production technology, although it can change the level of production capacity. It is worth noting that, as a consequence of operating leverage, the EBITDA rate has volatility which is higher the lower the level of $k$. This creates the incentive for equityholders to reduce capacity when the state is low.

We assume that capital depreciates both economically and for accounting purposes at a constant rate $d > 0$. So, given a capital stock $k$ and a debt level $b$, Earnings Before Taxes (EBT) are equal to the firm’s EBITDA minus depreciation and the interest to be paid for the outstanding debt:

$$y(k, b, \theta) = \pi(k, \theta) - dk - rb.$$

We introduce a corporate tax function, $g$, defined as a convex function of EBT, which we denote $y$, to model a limited loss offset provision:

$$g(y) = \begin{cases} y\tau_c^+ & \text{if } y \geq 0 \\ y\tau_c^- & \text{if } y < 0, \end{cases} \quad (3)$$

where $\tau_c^-$ and $\tau_c^+$, such that $0 \leq \tau_c^- \leq \tau_c^+ < 1$, are the marginal corporate tax rates for negative and positive earnings, respectively.\(^5\) Investors pay taxes on the returns on the securities issued by firms. We assume that taxes on cash flows are levied at constant rates: $\tau_e \in [0, 1]$ for equityholders, and $\tau_b \in [0, 1]$ for bondholders.

\(^4\)See also Gomes (2001), Hennessy and Whited (2005), Hennessy and Whited (2007), and Moyen (2007).

\(^5\)This choice for the corporate tax function is borrowed from Leland and Toft (1996). In unreported results we implemented also the more accurate convex tax function used by Hennessy and Whited (2005), with no substantial difference in our main conclusions.
We assume that the firms operations can be financed by issuing equity and debt and that financial markets rationally prices the cash flows paid to these securities.

As for debt financing, the companies issue risky callable consol bonds with face value $b \geq 0$ and a coupon rate equal, for practical convenience, to the risk free rate $r$. The debt adjustment cost function is

$$q(b', b) = \begin{cases} 
q_0 + q_1 |b' - b| & \text{if } b' \neq b \\
0 & \text{otherwise},
\end{cases}$$

where $q_0$ is a fixed component and $q_1$ is the issuance cost proportional to the change from current debt, $b$, to new debt $b'$.

The adjustment cost function, $q$, entails a cost in case of both an increment and a decrement of debt. This might be perceived as an oversimplification, because while issuing new debt ($b' > b$) entails underwriting costs, it is less clear what the debt retirement cost should be, as it is witnessed also by Leary and Roberts (2005), p. 2597. On the other hand, it is widely acknowledged that there are implicit costs, due to restrictions on the possibility to pay down debt in advance or on debt repurchase or due to secondary market illiquidity, that also affect the decision to reduce the level of debt.

Funds can be raised also by issuing equity. In this case, a flotation cost is incurred, which is motivated by information asymmetry and underwriting fees. Hence, if the amount raised by the firm is $cfe$, the actual (negative) cash flow by the equityholders is

$$cfe \cdot (1 + \lambda_1) - \lambda_0$$

where $\lambda_0 \geq 0$ is a fixed cost component and $0 < \lambda_1 < 1$ is a parameter defining the proportional flotation cost.

The dynamic framework is infinite-horizon and discrete-time. We assume that the firm has two control levers: the book value of assets in place, denoted $k$, and the face value of outstanding debt, $b$.

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6This function for direct costs for debt restructuring is different from the one in other models. Mauer and Triantis (1994) assume $q_1 = q_1^+$ when $b' > b$, and $q_1 = q_1^-$ when $b' < b$, where $q_1^-$ is in fact the cost for issuing equity, as in their model debt repurchase can be financed only by issuing new equity. In our model we do not have this restriction. Fischer, Heinkel, and Zechner (1989), Goldstein, Ju, and Leland (2001) and Strebulaev (2007), for analytic convenience (i.e., to preserve the scaling property), the adjustment cost is specified as $q(b', b) = q_1 b'$ if $b' \neq b$ and $q(b', b) = 0$ otherwise. That is, the cost is proportional to the total amount of debt issued, and not just to the increment $b' - b$. We do not need this simplification, because we solve the valuation problem using a numerical approach.

7This cost function is drawn from of Gomes (2001). Instead, Hennessy and Whited (2007) use a convex flotation cost function. In unreported results we implemented it, but with no qualitative change of our main conclusions.
B. Investment, dynamic capital structure, default

At any date, the firm can decide to invest or disinvest to reach a new level of assets $k'$. If there is positive investment, then the cost is represented by $\xi = k' - (1 - d)k$ and it can be financed either with internal funds, such as cash flows from operations, or with external funds, by issuing debt or equity. We assume that capital is homogeneous, so we cannot distinguish between capital additions made at different dates.

On the contrary, if the firm decides to disinvest, it does so at a liquidation price, and then the cash inflow is $\ell((1 - d)k - k')$, with $\ell \leq 1$. This introduces investment irreversibility in the model, and as a consequence, physical asset cannot be used as a substitute of cash.\(^8\) For notational convenience, to describe the payoff from investment/disinvestment $\xi$ we define the function $\chi(\xi, \ell)$ as

\[
\chi(\xi, \ell) = \begin{cases} 
\xi & \text{if } \xi \geq 0 \\
\xi \ell & \text{if } \xi < 0.
\end{cases}
\]

We model also the state of financial distress (liquidity crisis). If financial distress worsens, the firm defaults. Different conditions have been used in the literature to model financial distress. We assume that distress takes place when after-tax operating cash flow is insufficient to cover the coupon payment:

\[rb > \pi(k, \theta) - g(y(k, b, \theta)).\]

In this case, the firm sells at a discount $s \leq \ell$ the minimum amount of capital, $(rb + g - \pi)/s$, to make the promised payment.\(^9\)

Equityholders may decide to increase or reduce the debt to a new level $b'$ for the next period.\(^10\) We assume that bondholders do not have the power to block any additional debt issuance. In case it is optimal to change the level of debt to $b'$, all the outstanding

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\(^8\)This is different from what Hennessy and Whited (2007) and Moyen (2007) assume, because in their models the investment decision is fully reversible, or alternatively, capital is as liquid as cash.

\(^9\)While our definition of financial distress in line with Titman and Tsyplakov (2005), Streubulaev (2007), and Gamba and Triantis (2007), the consequences of financial distress are somehow different. A reduction of cash flow is directly introduced in Titman and Tsyplakov (2005) model. In fact, in their model financial distress generates a cost that is financed either by issuing debt or equity. In ours, the asset sale implies an additional investment that is financed exactly in the same terms: either with equity or debt. This is different also from Streubulaev (2007), because asset sales are based on a discount to market value in his model, as opposed to a discount to book value in our model. Asset sales motivated by liquidity crises have been documented by Asquith, Gertner, and Scharfstein (1994).

\(^10\)Leland and Toft (1996), Leland (1998), and Titman and Tsyplakov (2005), introduce a continuous retirement/reissuance rate to deal with (average) debt maturity in an infinite horizon setting. Although this may provide with an interesting additional viewpoint our analysis, we prefer not to introduce it here for simplicity.
debt is called at par value, \( b \), and new debt is issued at the market value \( D(k, b', \theta) \). While the assumption of calling at the face value preserves the rights of the current debtholders in the event the firm increases its debt level, on the other hand, it entails a refunding cost given by the difference between the market price and the face value of debt, \( D(k, b', \theta) - b \). Said difference will always be negative in case of a debt reduction, but it can also be positive when debt increases, and the market value of the new debt is higher than the face value of existing debt.

The firm can be taken off its steady state path in case of default on its debt obligations. We assume that all bonds have the same priority in this event. In the literature, many different mechanisms in which default can occur have been studied.\(^{11}\) In this work we assume endogenous default as the base case, because it is generally perceived as a more realistic description of the actual decisions made by firms.

In the event of default, we assume that debtholders can use debt collection laws to seize the residual value of the firm as an unlevered ongoing concerns net of bankruptcy costs.\(^{12}\) Hence, the debt tax shield is lost and absolute priority rule applies. Note that the unlevered asset value includes the positive value of the option to optimally lever the firm and decide the new investment policy. Moreover, to preserve the stationarity of the infinite horizon model, we assume that in case of default, after paying the bankruptcy costs, debtholders become the new owners continuing operations henceforth.

C. Security valuation

The valuation model can be described as a dynamic programming problem: at any date, after observing \( \theta \), for given \((k, b)\) the firm chooses a new level of capital, \( k' \), and a new level of debt, \( b' \). If the firm is solvent, investment and financing decisions can be made with no restriction. On the other hand, in case of default, we assume that no decision is made regarding investment and capital structure, and the equityholders exercise the limited liability right by surrendering the firm to bondholders at the current level of capital.

The value of the equity, denoted \( E(k, b, \theta) \), is the solution of a Bellman equation based on the optimization of the sum of current cash flow and the continuation value

\(^{11}\) They can be roughly divided into two main classes: exogenous default and endogenous default. The first is triggered by bondholders exercising a protective covenant in the bond indenture, i.e., default is determined simply by the state of nature and the contract, leaving no scope for any initiative. In general, default can be triggered at any exogenously specified value. On the other hand, the notion of endogenous default covers the situation where the default state is selected by equityholders so that the firm’s equity value is maximized. Under this approach, if equityholders do not find it worthwhile to provide the equity financing to cover debt services, they trigger default (i.e., equity becomes worthless).

\(^{12}\) This is the same as in Moyen (2007).
The value of debt rationally incorporates the optimal policy decided by equityholders. Hence, the current value of the debt is the present value of future payoff to bondholders contingent on equityholders (future) decisions.

According to the above settings, the cash flow to equityholders in case of no default, in state \((k, b, \theta)\) assuming the firm is solvent, for a decision \((k', b')\), is

\[
\text{cfe}(k, b, k', b', \theta) = \max \left\{ \pi(k, \theta) - g(y(k, b, \theta)) - rb, 0 \right\} + (D(k', b', \theta) - b) I_{b' \neq b}(b') - q(b', b)
- \chi \left( k' - k(1 - d) + \max \left\{ \frac{rb + g(y(k, b, \theta)) - \pi(k, \theta)}{s}, 0 \right\} \right), \tag{6}
\]

where \(I_{b' \neq b}(b')\) is equal to one when \(b' \neq b\) and zero otherwise, and where \(D(k', b', \theta)\) is the ex coupon price of debt, considering that the new book value of assets is \(k'\), and the new book value of debt is \(b'\).

Based on our assumptions on the economic environment, the actual cash flow to equityholders is

\[
e = \begin{cases} 
\text{cfe} \cdot (1 - \tau_e) & \text{if } \text{cfe} \geq 0 \\
\text{cfe} \cdot (1 + \lambda_1) - \lambda_0 & \text{if } \text{cfe} < 0.
\end{cases} \tag{7}
\]

The value of the equity at state \((k, b, \theta)\) is the solution of the dynamic program

\[
E(k, b, \theta) = \max \left\{ \max_{(k', b')} \left\{ e(k, b, k', b', \theta) + \beta E_{k, b, \theta} \left[ E(k', b', \theta') \right] \right\}, 0 \right\}, \tag{8}
\]

In this equation, the discount factor is \(\beta = (1 + r_z)(1 - \tau_e)^{-1}\), where \(r_z\) denotes the certainty equivalent rate of return on equity flows,\(^{14}\) and the expectation is computed with respect to the transition probability of the process in (1), conditional on the current state of the firm, \((k, b, \theta)\).

Interpreting Equation (8), at the current state, equityholders maximize their value, given by the current cash flow plus the continuation value, by selecting the new level of book value of asset and liabilities. In this case, the optimal policy is

\[
(k^*, b^*) = \arg \max_{(k', b')} \left\{ e(k, b, k', b', \theta) + \beta E_{k, b, \theta} \left[ E(k', b', \theta') \right] \right\}, \tag{9}
\]

\(^{13}\)We assume that managers always pursue equityholders’ interest. It is beyond the focus of our analysis to analyze such agency issues. In addition, as mentioned in the introduction, we assume perfect symmetric information between equityholders and bondholders.

\(^{14}\)The certainty equivalent rate of return on equity flows, \(r_z\), is determined under a tax equilibrium setting as \(r_z = r(1 - \tau_b)/(1 - \tau_e)\), where \(\tau_b\) is the personal tax on debt income and \(\tau_e\) is the personal tax on equity income. Notice that, in the same setting, the discount factor for bond flows is \(\beta_b = (1 + r(1 - \tau_b))^{-1}\). Hence, \(\beta_b = \beta = \beta_e\). See Sick (1990) for details.
provided that the value of equity is positive. If \( e(k, b, k^*, b^*, \theta) + \beta \mathbb{E}_{k, b, \theta} [E(k^*, b^*, \theta)] \) is negative (i.e., the firm cannot recover from financial distress), then equityholders default on servicing debt and surrender the firm to debtholders. In this case, the production capacity is kept at its current level, \( k(1 - d) \), and the firm becomes all-equity-financed: \( (k^*, b^*) = (k(1 - d), 0) \).\(^{15}\) To summarize, the optimal policy is

\[
\varphi(k, b, \theta) = \delta(k, b, \theta) \cdot (k(1 - d), 0) + (1 - \delta(k, b, \theta)) \cdot (k^*, b^*),
\]

where

\[
\delta(k, b, \theta) = \begin{cases} 
1 & \text{in case of default} \\
0 & \text{otherwise} 
\end{cases}
\]

is the default indicator function.

To compute \( D(k, b, \theta) \), the ex-coupon value of debt under the assumption that the firm is solvent, we have to determine the cash flow to debtholders at \( (k, b, \theta) \), when firm’s decisions are made:

\[
cfd(k, b, \theta, \varphi) = (1 - \delta(k, b, \theta)) \left[ rb(1 - \tau_b) + \mathbb{I}_{b' \neq b} (b') b + (1 - \mathbb{I}_{b' \neq b'} (b')) D(k', b', \theta) \right] \\
+ \delta(k, b, \theta) \min \{ E(k(1 - d), 0, \theta)(1 - c), b \},
\]

where \( (k', b') = \varphi(k, b, \theta) \) and \( E(k(1 - d), 0, \theta) \) is the value of the corresponding unlevered \((b = 0)\) firm from Equation (8), at the depreciated capital level \( k(1 - d) \), and \( c \) is the proportional bankruptcy cost.\(^{16}\) Hence, the value of debt is

\[
D(k, b, \theta) = \beta \mathbb{E}_{k, b, \theta} [cfd(k', b', \theta', \varphi)],
\]

Equations (8) and (13), determining the value of equity and debt, is a system of simultaneous non–linear equations that must be solved numerically. In Appendix A we describe the numerical method used in our research.

**D. Agency costs**

Endogenous investment and financing decisions creates agency issues that have an impact on debt price and capital structure decisions. There can be two distortions of the first

\(^{15}\)This is different from Mauer and Triantis (1994), who assume that an insolvent firm cannot make any decisions. Instead, in our model, if a firm is insolvent, it first tries to make up the debt payment using existing asset or issuing new equity. If still it is insolvent and equityholders do not find it profitable to refund, there is default.

\(^{16}\)Equation (12) sets also the debt principal as an upper bound for the cash flow to bondholders in case of bankruptcy. In unreported results, we investigated as alternative possibilities, the recovery cash flow to debtholders in case of default is either a proportion of the depreciated book value of asset, \( cfd(k, b, \theta) = k(1 - d)(1 - c) \), or of the face value of their claim \( cfd(k, b, \theta) = b(1 - c) \). The results presented in Section II are qualitatively not affected by this choice. We do not consider strategic debt service (see, for instance, Mella-Barral (1999)): it is beyond the scope of this paper to dissociate the events of default and liquidation.
best policy: one with respect to the optimal investment policy, and one with respect to
the financing policy. Although these two agency issues are strictly interconnected, for
clarity sake, we analyze them separately here below.

As for sub-optimality of investment policy, assuming that at the current state \((k, b, \theta)\)
the firm is solvent, debt financing may create an incentive for equityholders to deviate
from a firm value maximizing (first best) investment policy. In this case it either over-
invests (risk shifting/asset substitution) as proposed by Jensen and Meckling (1976), or
it under-invests (debt overhang), as explained by Myers (1977).17

Assume that the investment decision is \(k^*\) from (9). Given the decision \(b^*\), the
value of debt, \(D(k^*, b^*, \theta)\), rationally incorporates the effects of a deviation from a firm
value maximizing investment decision. If \(b = 0\), then the debt is issued at \(D(k^*, b^*, \theta)\)
and the equityholders bear the cost of the sub-optimal investment policy because they
collect less cash from debt issuance. If instead \(b > 0\), then equityholders have two
alternatives: they can decide either to change or to keep the current debt level. In
the first case \((b^* \neq b)\), the new bondholders rationally anticipate the equity-maximizing
investment policy, and they transfer this agency cost to equityholders through the price
\(D(k^*, b^*, \theta)\). Hence, equityholders immediately bear the consequences of their (second
best) investment decision through the refunding cash flow, \(D(k^*, b^*, \theta) - b\). Also in
the latter case \((b^* = b)\), the equity maximizing investment policy is incorporated into
the debt price although, differently from the previous case, bondholders cannot readily
transfer this cost to equityholders because the debt contract is long-term. Yet, they
anticipated this distortion from the first best decision when, in a previous year, the
equityholders decided to issue debt for a face value \(b\). So the price of debt reflects not only
current, but also future sub-optimal investment decisions. Therefore, the investment
decision, \(k^*\), is made by taking into account also the agency costs of sub-optimality of
the investment policy, which are eventually paid by equityholders. We can anticipate
that this agency cost is more severe in cases when it is optimal to invest \((k^* > k)\).

Despite the fact that they have received less attention, the effects of a sub-optimal
financing decision can be as important as the ones for the investment decision. This
agency cost is the value shortfall of the debt from the equity maximizing, as opposed to
the firm-maximizing, financing decision. Actually, different from other models proposed
to analyze the agency cost of sub-optimal investment, like Leland (1998), Childs, Mauer,
and Ott (2005), Moyen (2007), where the capital structure decision is made to maximize
the total firm value, in our model the leverage choice is aimed at maximizing equity
value.

17 Actually, we will show later in our analysis that current debt always creates an incentive to under-
invest. We almost never observe over-investment in our simulations. This is confirmed by Titman and
Tsyplakov (2005) and Sundaresan and Wang (2006) on a theoretical ground and empirically by Graham
and Harvey (2001).
In detail, assume that at \((k, b, \theta)\) the firm is solvent and that the financing decision, from (9) and for a given \(k^*\), is \(b^*\). If \(b^* \neq b\), the old bondholders receive \(b\). In principle, they do not suffer from sub-optimality of equityholders' capital structure decision because they receive more than the fair value of their claim, \(b \geq D(k^*, b, \theta)\). If instead \(b^* = b\), then the current bondholders bear the cost of sub-optimality (i.e. no change) of capital structure decision. While they cannot readily transfer this cost to equityholders, they rationally anticipates this in previous years when the debt was issued. So the price of debt is lowered by future second best capital structure decisions. This agency issue is more severe when \(k^* < k\). In this case, it would be optimal from a firm value maximization perspective to have the debt reduced, \(b^* < b\). Yet, as we will see, equityholders have no incentive to pay down the debt. Hence, the sub-optimal decision \(b^* = b\) creates an agency cost that is incorporated into the price of debt.

This distortion of the first best financing policy can be seen when investment is not irreversible. Differently from Titman and Tsyplakov (2005) (but similarly to Moyen (2007)), in our model the equityholders can partially reverse the investment decisions made in previous steps. This is an important feature of the model: when there is a downturn in the state variable, the equityholders have the incentive to voluntarily sell asset to smooth their cash flow over time, although the related cash inflow is not used to pay down debt. Because of this agency cost, the cost of debt is increased. Moreover, if this is done in the worst state, this increases also the default probability and the credit risk.

Note that, the distortion of the first best financing decision is absent when the debt maturity is prespecified (although optimally) and debt decisions are made after the firm is turn unlevered, as in Childs, Mauer, and Ott (2005),\(^{18}\) or when there is one-year debt, as in Hennessy and Whited (2007) and Moyen (2007).

To summarize, both the capital structure decision and the investment decision creates agency costs (due to sub-optimality with respect to first best) that are incorporated into the debt price or equivalently, on the yield. Hence, the optimal leverage decided by equityholders reflects a trade off between the tax shield on one side and the bankruptcy costs and agency costs on the other.

For this reason, by incorporating endogenous investment and financing decision in our structural model we can anticipate two related effects. On one hand, given the higher costs of debt due to agency issues we will find a lower leverage than predicted by structural models based solely on dynamic capital structure with constant production capacity. On the other hand, the agency issues imply also a higher yield on debt than other structural models also on high credit risk classes. In Section II we will see that these effects make the predictions of our model more in line with empirical observations.

\(^{18}\)In Childs, Mauer, and Ott (2005) the debt maturity is chosen to maximize the total firm value.
E. Restricted models

For comparison purposes, in our analysis we will consider also two restricted versions of the model described above.

The first restricted version is based on the assumption that there is no dynamic choice of investment and capital structure. Hence, we assume that debt and capital are kept constant over time. Since we are interested in a steady state solution, we assume that the physical capital of the firm is maintained at the level $k$ by forcing the firm to expense the depreciation, $dk$. Although there are many differences with the model proposed by Leland (1994) as far as the cash flow process, the tax environment and the financial market is concerned, this specification is in the same spirit of that model. For this reason, in what follows we will refer to it as the “Leland model”.

The second restricted specification of the model assumes static investment decisions, as in the “Leland model” described above, but allowing for dynamic decisions on capital structure. This specification is in the same spirit of Fischer, Heinkel, and Zechner (1989) or Goldstein, Ju, and Leland (2001), although some features of the cash flow process and the tax functions are different. In the rest of the paper we will denote this as the “FHZ model”, because of the resemblance with Fischer, Heinkel, and Zechner (1989).

Given the above restrictions, for convenience, we denote the full version (with dynamic investment and capital structure) as the “GAP model”, from the acronym of our names. This model is similar to Cooley and Quadrini (2001), although we have a richer specification for financial markets and taxes, and to Moyen (2007) and Hennessy and Whited (2007), but with the important difference of long-term as opposed to one-period debt. A model based on one-period debt omits the consideration of direct transaction costs and, most importantly, of refunding cost, which creates a liquidity risk (from the definition given by Childs, Mauer, and Ott (2005)) and eventually increases the default risk. Moreover, a model based on one-period debt would not account for agency costs due to a sub-optimal capital structure policy, as described above. Instead, in our model debt is long term.

Our model is different from Titman and Tsyplakov (2005) because

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19 Differently from Strebulaev (2007), we do not introduce financial distress in the version with static investment and dynamic capital structure. Since in this specifications there is no investment (apart from replacement of depreciated capital), allowing for fire sales would not permit to have a steady state solution of the problem, because eventually, all the capital stock would be depleted.

20 It should be noted that neither the “Leland” model nor the “FHZ” models include the agency issues presented by the full model.

21 A fourth possible version of the model would be the one with static debt and dynamic investment. This model has been studied by many authors, (Leland (1998), Childs, Mauer, and Ott (2005), Titman and Tsyplakov (2005), and Moyen (2007), just to mention a few of them) with the purpose of measuring the agency cost of the investment policy. We are not interested here in this case, because we want to analyze the effect that dynamic investment has on debt price when the capital structure is dynamic.
we deal also with investment reversibility and its effect on credit risk, whereas they assume that once new asset is in place, only depreciation can reduce it. As we will see, this feature has a large impact on the yield spreads.

To capture the agency cost dimension we will also consider two additional restriction of the base case (GAP) model: one uses only equity financing and hence the firm is constrained to stay unlevered ($b = 0 = b'$), and one constrained to have $k' \geq k(1 - d)$, with no disinvestment.

II. Analysis of credit risk

A. Determinants of credit risk

In this section, to analyze how credit risk depends on dynamic investment and capital structure decisions, we compare the value of debt in Equation (13) for the three specifications of the model (GAP, FHZ, and Leland) computed using the base case parameters from Table III.

Figure 1 plots the credit spread, $rb/D(k, b, \theta) - r$, as a function of the productivity of the firm’s capital stock, $\theta$. For the three versions of the model, the face value of debt is set at $b = 1.4$, which corresponds to the mode of the simulated distribution of debt face for GAP, and the value of capital stock is set at $k = 7.7$, which is the mode of the simulated distribution of capital for GAP. In Figure 1, we plot also the credit spread function for the GAP model when the investment policy is constrained to positive values for investment, $k^* \geq k(1 - d)$, and when the capital is fully reversible ($\ell = 1$). Although the figure is based on specific values for $k$ and $b$, the behavior described here below is general and does not depend on this choice.

The negative slope of the credit spread as a function of the productivity is as expected: highly profitable companies generate enough operating cash flows and have no problems to meet current debt obligations.

The credit spread for Leland is generally higher than the one for FHZ. This is because FHZ gives the possibility to adapt the capital structure over time. Since in FHZ the asset is constant (no investment), and debt is paid down at par when it is restructured, there is no agency cost of debt. So, the ability to dynamically adapt the capital structure, combined with the fact that callable debt must refunded at par, increases the value of debt. Put differently, there is no agency cost related to a sub-optimal (with respect to a firm value maximizing) capital structure policy. Also GAP has dynamic capital structure, but with the difference that in the latter we have also dynamic investment and so agency issues related to sub-optimal investment are present. In any case, FHZ
and Leland models are very close to each other, thus meaning that dynamic capital structure, per se, does not add much to the analysis of credit spreads.

The most remarkable fact from Figure 1 is that the credit spread for GAP is generally higher than for the two other models. This is due to the agency costs related to sub-optimal (with respect to a firm value optimizing) investment and debt policies, as described in Section I.D.

Analyzing first the agency issue due to investment decisions, the existing debt creates incentives to deviate from a firm value maximizing investment policy. Hence, the debtholders rationally anticipate the self-interested decisions made by equityholders reducing the value of debt. There are typically two types of incentives created by existing debt: over-investment and under-investment. We show that the reason for a higher yield spread in our model is actually under-investment.

Table IV shows the ratio $k^*/k$, averaged with respect to current $k$ at selected values for $\theta$, where $k^*$ is the optimal investment from Equation (9), for different levels of current debt face, $b$, at three possible states for firms productivity: low state ($\theta = 0.53$), average state ($\theta = 1$), high state ($\theta = 1.89$). The average ratio is lower than one (disinvestment) in low states, and higher than one (investment) in good states. In the average state, it is greater that one for low debt levels and almost one when $b \geq 1.76$.

First, for $b = 0$ we compare the ratio of the GAP model to the corresponding ratio for the constrained unlevered model, where we do not have any distortion of investment due to existing debt. While for low state the unlevered GAP firm and the constrained unlevered firm have almost the same policy, they behave differently when $\theta > 1$. In this case, we can see that the GAP firm invests more (i.e., the ratio $k^*/k$ is higher) than the constrained unlevered in case of an increment in productivity. This is because the constrained unlevered firm cannot issue debt to finance additional investment. When the GAP firm is heavily levered, it cannot issue additional debt, and so its investment ratio becomes almost equal to the one of the constrained firm. A second remark is that the ratio for the unlevered firm (either constrained or unconstrained) is lower than one (about 0.85 in the table) when productivity is low. This is evidence that the equityholders have an incentive to disinvest in worse states, which not motivated by the presence of debt.

When $\theta \geq 1$, the ratio is generally decreasing with respect to current debt. This is exactly the way debt overhang shows up in this model: because investment would benefit current debtholders (as explained in Section I.D), equityholders do not invest or invest less the higher the current debt. Remarkably, we never observe over-investment, that would be the case if the ratio was an increasing function of $b$.

The above analysis explains why in Figure 1 the credit spread for the GAP model constrained not to disinvest ($k^* \geq k(1 - d)$) is generally higher than the one predicted.

\textsuperscript{22}These observations are general and not restricted to Table IV, as confirmed by unreported analysis.
by FHZ and Leland. Notice that the yield spread in GAP is high also in good states because, when $\theta$ increases, new debt is issued to capture additional tax shield. Hence, also in good states the agency costs that affect debt price are increased accordingly. This fact is observed also by Moyen (2007).

The additional spread between the case constrained to $k^* \geq k(1-d)$ and the base case GAP model is due to the fact that when the productivity of the firm’s capital is low, it is optimal for equityholders to engage in disinvestment. By comparing the optimal investment policy of the levered GAP firm to the one of the corresponding constrained unlevered model, when $\theta < 1$ in Table IV, we can see that the levered firm generally disinvests more readily (i.e., the ratio $k^*/k$ is lower) than the constrained unlevered one. To say this differently, when the state is bad, the equityholders have the incentive to voluntarily sell part of the asset, and the higher is the current leverage, the bigger the incentive.

The motivation for selling capital stock at low states is twofold. First, in the attempt to smooth their intertemporal cash flow, when resources generated from operations are low, equityholders produce additional cash by selling physical capital at the liquidation price $\ell$, and they do not use it to pay down the debt. Second, the volatility of cash flow rate is increased, due to the operating leverage induced by fixed costs. To clarify this point, in Table V we compute the cash flow rate volatility, $\sigma_{CF}$, for three different capital levels, $k = 3$, $k = 7$, and $k = 11$, assuming that the current productivity is $\theta = 1$ and the current face value of debt is $b = 3$. The cash flow for a given $k$ and $\theta$, is

$$CF(k) = \pi(k, \theta) - g(\pi(k, \theta) - dk - rb) - rb,$$

where $\pi(k, \theta)$ is the EBITDA from Equation (2) and $g$ is the corporate tax function from (3). The growth rate of cash flow is defined as $CF(k, \theta')/CF(k, 1) - 1$, and $\mu_{CF}$ and $\sigma_{CF}$ are its expected value and standard deviation, respectively. From the last row of the table, we can see that the lower $k$ the higher $\sigma_{CF}$. Hence, operating leverage permits equityholders to better exploit the convexity of their value function. This creates a further incentive to disinvest in bad scenarios.

Under a first best capital structure policy, it would be optimal to reduce the debt ($b^* < b$) in case of disinvestment ($k^* < k$). Yet, as we will see later on, this would benefit the current bondholders but not the equityholders. For this reason, equityholders prefer to keep the debt constant ($b^* = b$), profiting from the cash flow generated from the asset sale. Since this sub-optimal debt policy is rationally incorporated into the debt price when the debt is issued, in Figure 1 we have a higher spread for this case.

In addition to the agency cost, from the debtholders’ perspective, disinvestment in bad states has the effect of decreasing the collateral (which is the value of the unlevered

\footnote{Note that this is the only case analyzed in Titman and Tsyplakov (2005).}

\footnote{Needless to say, cash flow rate volatility is even larger for lower values of current state, $\theta$.}
equity) in case of default, and of endogenously increasing business risk. In the end, debtholders rationally react by asking for the highest credit spreads. Yet, in our model (see Equation (10)) equityholders cannot sell assets to the point of causing the default, because in that case their policy is undone and the firm asset is restored.

To reinforce the above argument, in Figure 1 we plot also the credit spread as a function of productivity for the GAP model with fully reversible investment $\ell = 1$. We can see that the base case with $\ell = 0.75$ has always a lower credit spread than the reversible case. Reversibility increases the incentives to disinvest in bad states, thus increasing the impact of the agency cost due to a sub-optimal debt policy and of default risk.

For illustrative purposes, in Figure 2 we plot the patterns of productivity, $\theta_t$ (multiplied by 10 to increase visibility), asset, $k_t$, face value of debt, $b_t$, equity value, $E(k_t, b_t, \theta_t)$, and ex-coupon debt value, $D(k_t, b_t, \theta_t)$ over a time span of 100 years. The patterns are obtained by applying the optimal solutions for the three models (Leland, FHZ and GAP) to the same scenario for the exogenous state variable, $\theta_t$. In GAP, the level of capital stock is endogenously determined. Equityholders decide in their best interest how often and how much to invest and disinvest and how often to change the capital structure. In FHZ, only the capital structure decision is endogenous and the asset is fixed at $k = 7.7$. In Leland, both the asset and the debt are constant at values $k = 7.7$ and $b = 1.4$, respectively.

While we know that the observation of a specific scenario does not permit to derive any general conclusions, it is interesting to notice that in GAP and FHZ, the debt policy is very stable, because the equityholders do not have incentive to reduce the level of debt. Voluntarily reducing the level of debt would entail a cost for them with the effect of benefiting bondholders, who are paid their claim at par. For this reason debt reductions are infrequent (but possible) and equityholders prefer to default instead of reducing leverage as a consequence of a reduced productivity. Overall, this increases the agency cost of debt due to a sub-optimal capital structure policy.

As for the investment policy, even if in GAP capital is partly irreversible (for the base case, 25% of its value is lost in the event of a disinvestment and 40% in case of forced liquidation), the company is active on the investment side i.e., the book value of assets is frequently above the level in the other two models for large values of cash flow, and in addition, investment/disinvestment decisions are often made. Yet, investment is not as ready when there is an upturn of $\theta$ as disinvestment is ready to follow a downturn.

To illustrate this fact, from a simulated sample created using the base case parameters following the procedure described in Section II.B, for those companies active and levered at the beginning of each period, we compute the ratio of number of periods when there is a simultaneous increase in $k$ (i.e., $k_t \geq k_{t-1}$) and in $\theta$ (i.e., $\theta_t > \theta_{t-1}$), over the number of

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25This is the only case considered in Hennessy and Whited (2007) and Moyen (2007).
periods when there is an increase in $\theta$. In 25% of the cases of an upturn in $\theta$, the optimal decision is to invest. Analogously, for active and levered companies at the beginning of each period, we compute the ratio of the number of times a downturn in $\theta$ (i.e., $\theta_t < \theta_{t-1}$) triggers a decrease in $k$ (i.e., $k_t < k_{t-1}$) over the number of periods $\theta$ decreases. When $\theta$ worsens, the chances of disinvestment are about 43%, thus confirming that equityholders over-react by reducing (or not replacing depreciated) capital more frequently. As we saw before, this generates the additional agency issue of a suboptimal capital structure policy, because the debt is not reduced accordingly.

The end result is that the GAP model is characterized by a lower average value of debt than FHZ. Notice also that for the specific scenario we selected in Figure 2, it is optimal for the GAP firm to stay occasionally unlevered, and the duration of the unlevered state is longer than for the FHZ case. Again, this effect is due, *coeteris paribus*, to the higher cost of debt.

**B. Simulation procedure**

The goal of what remains of this section is to compare the ability of the GAP model to match the empirical measures of credit risk (the credit spread, the leverage, and the default rate), relative to the restricted versions of the model that we named FHZ and Leland. To this aim, we describe the simulation procedure which is used to calibrate the model.

We create three samples of firms, one for each specification of the model (GAP, FHZ and Leland) described in Section I.E. The samples are obtained by simulating 20,000 paths for the state variable $\theta$ using the stochastic model in Equation (1) for 130 steps (years). Next, we apply the optimal policy $\varphi$ from Equation (10) found by solving the dynamic program for the relevant specification. At every step, along each path, the realization of the exogenous state variable, combined with the current debt level and stock of capital provides the endogenous values of equity, debt and credit spread as determined by the optimization problem. To get rid of the influence of the initial condition, we drop the first 30 steps.

In case of default, the bondholders become the new equityholders and are given the unlevered asset value as an in-kind payment. Since we want to hold constant the size of the sample over time, the bondholders keep the EBIT-machine in operations by paying the bankruptcy costs using their own cash. After default, the firm is revived. The state after bankruptcy is different for the three models.

As for the Leland model, since it has constant capital stock $k_0$ and debt level $b_0$, after default, the firm is revived at the initial capital stock $k_0$ and with the leverage $b_0$. 
For the FHZ model, given the constant capital stock $k_0$, after defaults, the new equityholders start operations at the current level of capital $k_0$, and then they may relever the firm applying the optimal policy found in the dynamic program.

Lastly, for the GAP model, in case of default at time $t$, the new equityholders carry on operations for the given capital $k_t$, pay bankruptcy costs and, in case, optimally issue new debt. For a particularly low level of capital stock, the firm stay unlevered and will never find optimal to restart operations and would be dropped from the sample. To keep the size of the sample of active firms constant, we assume that a lower bound for capital, denoted $k_d$, is set in our simulation so that the condition $\Pr\{\pi(k_d, \theta) > dk_d\} > 0$ holds true. This means that there is a positive probability that the EBIT will be positive in the next step and the firm is restarted.\(^{26}\)

The goal of simulation is to obtain a mapping of a credit risk measures on quasi-market leverage.\(^{27}\) Actually, in our model the leverage at a given date is path-dependent because it is determined both by the exogenous state variable and by the sequence of past decisions. So, as it will be clarified later, it is possible to determine a mapping between quasi-market leverage and the credit risk measures only by explicitly considering the pattern of past decisions; i.e., by using Monte Carlo simulation. Once the desired mapping is obtained, we compare the simulated risk metrics from the models to the empirical ones. This comparison is at the basis of our calibration procedure as described in Section II.C.

**C. Model calibration**

Our sample data is from Compustat. We exclude financial, insurance and real estate firms (SIC code 6000-6999) and also regulated utility firms (SIC code 4900-4999). Second, we drop any firm-year observation with (i) non listed share price in the Center for Research on Security Prices (CRSP)/Compustat Data Merged files, (ii) non available Standard & Poor’s Long Term Issuer Credit Rating or (iii) any missing data for the variables considered. We end up with an unbalanced panel of firms from years 1997 to 2005 with between 887 and 1110 companies per year (in total the data set has 9048 observations).

\(^{26}\)Alternatively, we can say that the state $(k, b, \theta)$ with $k = k_d$ is not absorbing. In our simulations, we set $k_d$ such that $\Pr\{\pi(k_d, \theta) > dk_d\} = 0.5$.

\(^{27}\)The reason for our choice of quasi-market leverage as the explanatory variable for the credit risk measures is motivated by the way firms are categorized into different risk classes, as explained in Section II.D. The credit risk measures we consider are the credit spread on corporate debt and the default rate.
Data variables used in the computation of company metrics are: book value of common equity (Compustat item 60), book value of long-term debt (item 9), book value of short-term debt (item 44), earnings before interest and taxes or EBIT (item 178), interest expenses (item 15), fiscal year end share price (CRSP item PRCC12) and, finally, number of outstanding shares (CRSP item CSHOQ12). The firms’ metrics built upon these variables and their definitions are: (1) quasi-market leverage, defined as total debt (long-term plus short-term) over total debt plus the product of market price of share and number of outstanding shares; (2) total debt over EBIT; (3) interest coverage, defined as EBIT over interest expenses; and (4) return on assets (ROA) computed as EBIT over total debt plus the product of market price of share and number of outstanding shares.

Next, based on the S&P’s Long Term Issuer Credit Rating, we sort companies in seven (whole letter) risk classes, ranging from AAA/Aaa (companies with extremely strong capacity to meet financial obligations, code 2) to C (class made of companies with rating CCC/Caa or worse; i.e., code 17 and higher). The classification procedure is described in subsequent Section II.D. Within each risk class, from the panel data we compute the mean of each metric relevant for our analysis.

The three models are calibrated on average credit spreads and also on empirical average annual default rates, both measured per (whole letter) credit merit class. The average annual default rates are obtained as

\[ 1 - \exp \left( \frac{1}{10} \cdot \log(1 - DR_{10}) \right) , \]

where \( DR_{10} \) is the Moody’s 10 year default rate (Average issuer-weighted corporate percentage default rates by whole letter rating, 1983-2005. Moody’s Investors Service (2006)). In particular, the 10 year default rate we used are: 0.208 for AAA, 0.415 for AA, 1.248 for A, 4.721 for BBB, 21.038 for BB, 46.931 for B, and 78.673 for C. The average credit spreads paid by industrial bonds in each specific rating class are obtained from Reuters and are referred to year 2004. All the above results are summarized in Table I.

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28Alternatively, it is total assets (item 6) less total liabilities (item 181) less preferred stocks (item 10).

29Total debt over EBIT, the interest rate coverage, and ROA are used in the classification procedure described in Section II.D.

30Originally, our data time period was 1995-2005. As a stability check, we split the sample into two subsamples (1995-1999, and 2000-2005) and computed the median for each subsample. We also carried out the analysis on a yearly basis. On this last analysis, we noticed that for years 1995 and 1996 companies in class C were very few, and more importantly, a great number of them had a zero figure for long-term debt in the data base. To avoid distortions, we decided not to include these two years in the sample.

31We checked also other sources of information on credit spreads and cumulative default rates, with no substantial difference with respect to the figures reported in Table I.
Our analysis relies also on previous papers and some of the parameters we use are derived from them. Table III reports the base case value of model parameters. Specifically, the value of the parameters of the flotation cost function, $\lambda_0 = 0.08$ and $\lambda_1 = 0.03$, are derived from Gomes (2001) and are in line also with the ones in Strebulaev (2007). Proportional debt adjustment costs are provided in Fischer, Heinkel, and Zechner (1989) with $q_1 = 0.01$. Since we will exploit the adjustment costs in the calibration procedure described below, we will consider $q_1 = 0.01$ as our starting point.

We use a return to scale of $\alpha = 0.47$, higher than the one in Gomes (2001), who proposes $\alpha = 0.30$ but with a different hypothesis on productive technology. We include also a fixed cost, to increase both the operating and (as a consequence) the financial leverage. Gomes (2001) also provides $\rho = 0.62$ and $\sigma = 0.15$, that are lower than ours. In Moyen (2004), $\alpha = 0.45$, $\rho = 0.6$ and $\sigma = 0.2$ and the fixed cost is $F = 1.3$. So, our parameters are in line with hers, with the exception of a higher $\rho$, which will be used in our case to span the distribution of credit spread in the calibration procedure. Note that Hennessy and Whited (2005) provide a (structural) estimate of $\rho$ equal to 0.74, not far from our choice. The depreciation rate, $d$, and the salvage value, $s$, are the same as in Hennessy and Whited (2005). The selling price of capital stock in case of disinvestment, $\ell = 0.75$, represents the degree of investment irreversibility and is an important determinant of credit risk. We take it from Gamba and Triantis (2007), but in any case we did sensitivity on $\ell$, to make sure that our results are not driven only by this feature of the model.

The personal tax rates $\tau_e = 0.15$ and $\tau_b = 0.25$ are in line with the ones in Hennessy and Whited (2007) and in many other papers. The corporate tax rate is midway between the one in Fischer, Heinkel, and Zechner (1989) ($\tau_c^+ = 0.50$) and the one in Hennessy and Whited (2007) ($\tau_c^+ = 0.40$). With our selection of parameters, the rate of net tax advantage to debt is 0.3324. The tax rate for losses is 5%, which means very limited possibility of loss offsetting. In our numerical experiments, the last parameter has only a second order effect on the simulated statistics. Finally, the risk-free rate is 5% on an annual basis.

We use corporate taxes $\tau_c$, volatility $\sigma$, the return-to-scale parameter, $\alpha$, the production fixed cost, $F$, fire sales discount, $s$, liquidation price, $\ell$, the equity flotation costs, $\lambda_0$ and $\lambda_1$, and the debt adjustment costs, $q_0$ and $q_1$, to calibrate the three models to empirical data.

The calibration of each of the three specifications of the model is based on fitting the empirical credit spreads per risk class from Table I, with the corresponding credit spreads from a simulated sample generated from the model. Differently from previous authors (e.g. Eom, Helwege, and Huang (2004) and Ericsson, Reneby, and Wang (2005)), our calibration procedure is based on the attempt to fit not only an average measure of credit spread for investment and speculative grade bonds. We are aiming at matching both the distribution of firms into different credit classes and the credit spread in each
class because we think this procedure can capture the heterogeneity of firms in different risk classes.

For a given simulated firm, we compute the credit spread, \( rb/D(k, b, \theta) - r \), where \( D(k, b, \theta) \) is the ex coupon price of corporate bonds. We compute the sample average of the credit spreads in each rating class and then we generate the distribution of credit spreads produced by the specification of the model with the given choice of parameters. An example is offered in Table VII.

Generally speaking, in our analysis we found very difficult for the three specifications of the model to match perfectly the credit spreads paid by the class C (unless the overall fitting is worsen). This class gathers all the simulated and actual firms that are not assigned to the other classes. In fact, it includes both highly speculative grade bonds and bonds essentially in default. For this reason, our effort has been concentrated mostly in fitting the classes from AAA to B.

For Leland model, differently from the other two specifications, we simulated the sample using a uniform distribution of initial face value of debt. This has been done with the purpose of having a significant number of simulated firms in all credit classes.

In our experiments, we noticed that the parameters related to the production function, \( \alpha, F, d \), and to the exogenous state variable, \( \rho \) and \( \sigma \), are the ones that have the largest impact on the distribution of firms among the credit classes and hence of the distribution of credit spreads. To a lesser extent, the distribution of firms among the credit classes depends also on the rate of net tax advantage to debt

\[
\tau^+_c \left(1 - \frac{1 - \tau_b}{1 - \tau_e}\right),
\]

on the liquidation price, \( \ell \), and on the salvage value, \( s \). Finally, the other parameters, like the equity flotation costs, \( \lambda_0 \) and \( \lambda_1 \), the debt adjustment costs, \( q_0 \) and \( q_1 \), and the bankruptcy cost, \( c \), have an even smaller impact on the distributions, and are used mostly to improve the fitting of simulated credit spreads.

Lastly, model parameters are chosen also in order to make unlikely to have optimally unlevered firms in the sample.

**D. Classification of simulated firms into credit classes**

The most critical part of the calibration procedure is the classification of simulated firms into the whole letter classes from AAA to C.

In order to find the best sorting criterion, we run several ordered probit models on our Compustat sample. In all of them, the dependent variable is the S&P’s seven (whole
letter) risk classes discussed previously. It takes values from 1 (class AAA) to 7 (class C).
We have chosen an ordered qualitative response model like the probit model because the
credit risk class is an inherently multinomial choice that merely represents an ordering.

As potential independent variables (i.e., as sorting criteria), we have selected the
following metrics: book-leverage (debt over debt plus book value of equity), quasi-
market-leverage (debt over debt plus the product of market share price and number
of outstanding shares), debt over EBITDA, debt over EBIT, EBITDA over interest
expenses, EBIT over interest expenses, and finally ROA (EBIT over total debt plus the
book value of equity).

In unreported analysis, we estimated the model using each of the potential criteria
as the sole independent variable. We observed that the independent variable is always
statistically significant except for EBITDA and EBIT over interest expenses. However,
when book leverage together with quasi-market-leverage is considered, the first variable
is no longer significant. Quasi-market-leverage then carries the same information on the
credit worthiness of a firm as book leverage. As a result, we dropped book leverage from
the final model specification. Similarly, when both debt over EBITDA (or, alternatively,
debt over EBIT) and quasi-market-leverage are introduced, the first variable becomes
not significant. Again, the same conclusion follows.

Our final specification (Table II) only includes quasi-market leverage and ROA as
independent variables. The large $t$-ratio on quasi-market leverage lead us to conclude
that this metric constitutes the primary credit risk sorting criterion. As such, we clas-
sify a firm by minimizing the distance between the simulated quasi-market leverage,
$b/(E(k, b, \theta) + b)$, and the same measure, specific for each rating class, from Table I.

E. Distributional properties of credit risk measures

The aim of this section is to compare the three models on the basis of their ability to
match the empirical measures of credit risk introduced in Section II.C. To this aim, we
analyze how closely the mapping between leverage and each measure of credit risk fits
the mapping obtained for the sample data and summarized in Table I.

The novelty of our analysis lies in the fact that we do not calibrate the model to
match the average credit spread of the complete data set. Rather, our goal is to match
the average credit spreads per risk class to properly account for heterogeneity of firms in
different risk classes. This choice permits us to jointly test the descriptive properties of
our model vis-a-vis more simplified models. As it is explained in Section II.C, we classify
the simulated companies into the whole letter classes from AAA to C and we compute
the sample mean of the credit spreads in each rating class. Finally we generate the
distribution of credit spread produced by the specification of the model. Besides credit
spreads, we estimate also the default rate as a measure of credit risk, and compare it to empirical default rates on a per class basis.

Table VII presents the estimated sample average (together with the standard error) of the credit spread in each rating class for each model, vis-a-vis the empirical values, that we report from Table I for convenience. For classes from AAA to BB, the GAP model provides estimates in line with Leland model, whereas the FHZ model fails in the AAA class. For risk classes B and C, the GAP model approximates empirical credit spreads remarkably better. That is, GAP model has a clear advantage for the worse risk classes. We note that for C class, the credit spread estimated from FHZ model is lower than the one estimated from Leland model. Thus, dynamic capital structure alone does not seem to be enough to capture the empirical increase in credit spreads. However, the inclusion of endogenous investment, almost doubles the credit spread with respect to FHZ model in worse risk classes. As discussed above, this is due to the contribution to credit spread of agency costs due to investment and financing decisions made in the interest of equityholders. This induces higher yield spreads than FHZ also for AAA firms. To ease the inspection, in the same table we plot the average credit spreads per risk class.

The discussion of Table VII, and of the subsequent tables, should be matched with Figure 3 and the companion Table VIII. This figure includes both the estimated risk class distribution and the empirical risk class distribution for GAP and FHZ, built according to Section II.C. Our aim is not only to match the credit measures but also to fit as much as possible the empirical risk class composition/profile.\textsuperscript{32} The risk class distribution for the GAP models, though slightly skewed to the left, resembles more the empirical distribution than the risk class distribution for FHZ model, which is bi-modal. We consider this result as evidence in support that the model has a greater ability to match the actual firm characteristics, also with respect to credit ratings. In addition, the number of companies that FHZ places in risk class C (around 20\%) is in strong contrast with what we observe in practice (only 2.34\% are in C class or below).

Strictly related to the above described effect is the remark from Table VIII, that FHZ and Leland models predicts a leverage respectively of 68\% and 72\% for class C, which is higher than predicted by GAP (61\%) and much higher than empirically observed (57\%). This fact is in line with the well documented evidence that structural models with dynamic capital structure predict much higher leverage than it is empirically observed.

We compare the per-year average default rates generated by the three models to the empirical ones. The average one-year default rate in Table IX are obtained by averaging across all the years of the simulated sample the default rates calculated as number of defaults in each year over the number of non defaulted levered firms at the beginning of the period.

\textsuperscript{32}The histogram is meaningful only for FHZ and GAP, since for Leland the distribution of firms among risk classes is (artificially) generated to populate all the risk classes, which cannot be spanned by a static debt level.
of the year. This is done on a per rating class basis. The simulated default rates are compared to the equivalent average annual default rate obtained from Table I. While the three models fit the default rate in high credit risk classes (BB through C), the prediction of FHZ and Leland models is remarkably poor for low risk classes. For the same classes GAP seems to provide a fair prediction of the default rate. These remarks reinforces the argument that a structural model based solely on dynamic capital structure is an inappropriate setting for explaining the default risk.

III. Concluding remarks

We have developed a dynamic structural model for credit risk of corporate debt featuring endogenous investment, endogenous capital structure, and endogenous default. Investment and leverage policy are simultaneously decided to maximize equity value. Corporate investments are partially irreversible and are financed with current cash flow as well as with bond and share issuance. The firm faces constant and proportional debt adjustment costs and equity flotation costs. In case of financial distress, fire sales of assets are made at a discount. Likewise, in case of bankruptcy, bondholders pay bankruptcy costs. The corporation has a convex tax function, thus including a limited loss offset provision, and investors face personal taxes, with higher rates for bond flows than for equity flows. To analyze the joint effect of debt adjustment costs, of agency costs due to the investment and financing policy, and of default risk, we use a risky consol callable bond.

The model is a discrete-time infinite horizon Markov Decision Process, and the solution must be found numerically. The related Bellman operator is two-dimensional, because we value debt and equity simultaneously. For this reason we introduced an algorithm to solve it which, to the best of our knowledge, is new in this field.

Our goal is to analyze the effect of endogenous investment decisions on the price of debt and default protection, describing the extent to which agency costs affect the yield spread. To do this, we compared the results to several restricted versions of the general model.

We calibrated the model using firm accounting information from Compustat North America Industrial Annual and market data from the Center for Research on Security Prices (CRSP), from Moody’s Investor Service for the cumulative default rates, and Reuters for the credit spread paid by industrial bonds.

Using an approach based on simulated samples of firms, we first classified the firms in rating classes using the quasi-market leverage as the discriminatory variable. We then compared different credit worthiness metrics produced by the model to the empirical
ones. In particular, we compared credit spreads, the rating class distribution, and the average default rate.

Our results show that endogenous investment is a major determinant of capital structure decisions and hence of credit risk, because of the inherent agency costs related to sub-optimal (with respect to a firm value maximizing) investment policy and capital structure policy. Moreover, the introduction of endogenous investment significantly improves the performance of a structural model in predicting empirical data on credit risk on corporate debt.

On the other hand, a structural model with dynamic capital structure has only limited explanatory power on credit risk because the change of debt to exploit the tax shield is made in a limited number of cases and/or the debt is not the preferred instrument for the firm to raise funds. Overall, we observe that the ability to dynamically adapt the capital structure, per se, has less impact on credit risk than dynamic investment.
A. Numerical solution of the fixed point problem

In this section we describe the numerical procedure we use to solve the complex dynamic programs introduced in Section I. For the sake of brevity, here we will refer only to the more general model (GAP).

The general valuation model for equity and debt presented in this paper belongs to the class of continuous decision infinite horizon Markov Decision Processes. The solution method is based on successive approximations of the fixed point solution of the valuation operator using a dynamic programming approach. For numerical purposes we apply this method to an approximate discrete state-space and discrete decision valuation operator.\footnote{See Rust (1996) or Burnside (1999) for a survey on numerical methods for continuous decision infinite horizon Markov Decision Processes.}

Gauss-Hermite quadrature method (see Tauchen (1986)) is used to approximate the dynamics of the logarithmic AR(1) process in (1) with a finite state Markov chain. We denote \( y = \log(\theta) \), and take \( s \) discrete abscissae in an interval of semi-width \( I_p = 4\sigma/\sqrt{1 - \rho^2} \), and centered on the long term mean of process, \( \eta/(1 - \rho) \). The set of the discretized state variable is \( \tilde{Y} = \{ \tilde{y}(s) \mid s = 1, \ldots, S \} \), where

\[
\tilde{y}(s) = \frac{\eta}{1 - \rho} - \max \left\{ \left( \frac{S - 1}{2} + 1 \right) - s, 0 \right\} u + \max \left\{ s - \left( \frac{S - 1}{2} + 1 \right), 0 \right\} u,
\]

with \( u = 2I_p/S \).

Next, we define the cells for the state variable as \( c(j) = [Y(j), Y(j + 1)] \), for \( j = 1, \ldots, S \), where

\[
Y(1) = -\infty, \quad Y(j) = \frac{\tilde{y}(j) + \tilde{y}(j - 1)}{2}, \quad j = 2, \ldots, S, \quad Y(S + 1) = +\infty.
\]

The transition probability matrix (under the risk-neutral measure) is given by the probability, conditional of the current state \( y \), that the future state is \( y' \). Given the above approximation, this is equivalent to the probability \( \Pi(i, j) \) that \( y' \) falls into cell \( c(j) \), given the current state \( y = \tilde{y}(i) \), for all \( j = 1, \ldots, S \) and all \( i = 1, \ldots, S \):

\[
\Pi(i, j) = \Pr \{ y' \in c(j) \mid y = \tilde{y}(i) \} = N \left( \frac{Y(j + 1) - \eta - \rho \tilde{y}(i)}{\sigma} \right) - N \left( \frac{Y(j) - \eta - \rho \tilde{y}(i)}{\sigma} \right).
\]

The transition probability matrix is \( \Pi = (\Pi(i, j), i, j = 1, \ldots, S) \). In our computation we use the values \( \tilde{\theta} = \exp(\tilde{y}) \), collected in the set \( \tilde{X} \), with the transition probability matrix \( \Pi \).
We set the upper bound for capital stock, $k_u$, and for the face value of the debt, $b_u$ respectively, in a way that they are never binding for the optimization problem.\footnote{Specifically, for the optimal set of parameters in Table VI, we use $k_u = 13$ and $b_u = 6$ for the three specifications of the model.} Next, we discretize the intervals $[0, k_u]$, and $[0, b_u]$ into $N_k$ and $N_b$ values respectively. We denote $\tilde{K}$ the discretized set for capital stock, defined as

$$\tilde{K} = \left\{ k_j = k_u (1 - d)^j | \ j = 1, \ldots, N_k \right\} ;$$

(14)

$\tilde{B}$ the set of discrete values for debt. The set $\tilde{B}$ is obtained by taking equally spaced values of face value of debt. To accelerate the convergence of the iterative procedure, we proceed by solving a sequence of discretized versions of the problem, each based on an increasingly denser grids in $[0, k_u] \times [0, b_u]$.

For a given $N$, at the $n$-th stage of this procedure, $n = 1, 2, \ldots, N$, we discretize the intervals $[0, k_u]$, and $[0, b_u]$ into $N_k^n$ and $N_b^n$ equally spaced values respectively,\footnote{To keep the structure described in (14), we equally space the logarithm of capital stock.} with $N_k^n > N_k^{n-1}$ and $N_b^n > N_b^{n-1}$, where $N_k^0$ and $N_b^0$ are prespecified. We denote $\tilde{K}^n$ and $\tilde{B}^n$ the discretized set for capital and debt, respectively, and to keep notation simple, we denote $(k, b)$ as the discretized control variable. Hence, at each stage the controlled state space is $\tilde{K} \times \tilde{B}$ and has size $N_k^n \cdot N_b^n$.

At stage $n$, we solve the problem

$$E(k, b, \theta) = \max \left\{ \max_{(k', b')} \left\{ e(k, b, k', b', \theta) + \beta \mathbb{E}_{k, b, \theta} [E(k', b', \theta')] \right\} , 0 \right\} ,$$

where function $e(k, b, k', b', \theta)$, defined in Equation (7), depends also on $D(k, b, \theta) = \beta \mathbb{E}_{k, b, \theta} [\text{cfd}(k', b', \theta', \varphi)]$ and

$$\text{cfd}(k, b, \theta, \varphi) = \begin{cases} \begin{align*} rb(1 - \tau_b) + \beta \mathbb{E}_{k, b, \theta} [D(k^*, b, \theta')] & \text{ if } b = b^* \text{ and } E(k, b, \theta) > 0 \\ rb(1 - \tau_b) + b & \text{ if } b \neq b^* \text{ and } E(k, b, \theta) > 0 \\ \min \left\{ \max \left\{ E(k, 0, \theta)(1 - c) , 0 \right\} , b \right\} & \text{ if } E(k, b, \theta) = 0 \end{align*} \end{cases}$$

for all $(k, b, \theta) \in \tilde{K} \times \tilde{B} \times \tilde{X}$.

More succinctly, the fixed point problem at stage $n$ is

$$E = \Gamma^n(E, D)$$
$$D = \Psi^n(E, D)$$

(15)

where $\Gamma^n$ and $\Psi^n$ denote the approximate operators, when we have $N_k^n \cdot N_b^n$ discrete points in the control set.

\[\text{\scriptsize \text{To keep the structure described in (14), we equally space the logarithm of capital stock.}}\]
The fixed point solution of the system of non-linear equations (15) is found by successive approximations. This means that, given the guesses $E^n_0$ and $D^n_0$, we iterate the following

\begin{align*}
E_j &= \Gamma^n(E_{j-1}, D_{j-1}) \\
D_j &= \Psi^n(E_{j-1}, D_{j-1})
\end{align*}

until convergence.

For equity value, we define the McQueen-Porteus error bounds (see Rust (1996))

\begin{align*}
\varepsilon(E) &= \frac{\beta}{1 - \beta} \min \{E_{j+1} - E_j\} \quad \mathcal{E}(E) = \frac{\beta}{1 - \beta} \max \{E_{j+1} - E_j\}
\end{align*}

and $\varepsilon(E) = |\mathcal{E}(E) - \varepsilon(E)|$. We define also

\begin{align*}
\varepsilon(D) &= \frac{\beta}{1 - \beta} \min \{D_{j+1} - D_j\} \quad \mathcal{E}(D) = \frac{\beta}{1 - \beta} \max \{D_{j+1} - D_j\}
\end{align*}

and $\varepsilon(D) = |\mathcal{E}(D) - \varepsilon(D)|$. In our computations we repeat the procedure until $\varepsilon(E) \lor \varepsilon(D) < 10^{-8}$. We denote $(E^n, D^n)$ the (approximate) fixed point solution of the system of non-linear equations (15). Once we have $(E^n, D^n)$, we determine the initial guesses $(E^{n+1}_0, D^{n+1}_0)$ for the $(n + 1)$-th stage by interpolation.

In our computational experience, this approach to obtain the discretized solution with $N_k^N \cdot N_b^N$ points and for the desired accuracy is much faster than the direct application of successive iterations to the fixed point problem

\begin{align*}
E &= \Gamma^N(E, D) \\
D &= \Psi^N(E, D).
\end{align*}

Once we have found the optimal solution for the fixed point problem of valuing debt and equity, we can determine the optimal policy $\varphi(k, b, \theta)$ by looking for the arg-max of equity at the discrete states $(k, b, \theta)$.

We solve the model using $S = 21$ points for $\bar{X}$, $N_k = 71$ points for $\bar{K}$ and $N_b = 35$ points for $\bar{B}$. 

30
References


<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>%</th>
<th>CS (bp)</th>
<th>QML (%)</th>
<th>DR (%)</th>
</tr>
</thead>
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<tr>
<td>AAA</td>
<td>69</td>
<td>0.76</td>
<td>55</td>
<td>6.29</td>
<td>0.02</td>
</tr>
<tr>
<td>AA</td>
<td>211</td>
<td>2.33</td>
<td>65</td>
<td>8.50</td>
<td>0.04</td>
</tr>
<tr>
<td>A</td>
<td>1415</td>
<td>15.64</td>
<td>92</td>
<td>16.12</td>
<td>0.13</td>
</tr>
<tr>
<td>BBB</td>
<td>2652</td>
<td>29.31</td>
<td>149</td>
<td>26.01</td>
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<tr>
<td>BB</td>
<td>2705</td>
<td>29.90</td>
<td>234</td>
<td>34.74</td>
<td>2.33</td>
</tr>
<tr>
<td>B</td>
<td>1784</td>
<td>19.72</td>
<td>675</td>
<td>43.38</td>
<td>6.14</td>
</tr>
<tr>
<td>C</td>
<td>212</td>
<td>2.34</td>
<td>1500</td>
<td>57.19</td>
<td>14.32</td>
</tr>
<tr>
<td>total</td>
<td>9048</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table I: **Statistics from the dataset.** Source: Compustat (1997-2005) (Compustat North America Industrial Annual files. Financial, insurance and real state firms (SIC code 6000-6999) and regulated utility firms (SIC code 4900-4999) are excluded), Reuters (2004) and our computations from Moody’s Investor Service (2006). N is the number of firms/years. CS is the average credit spread by whole letter class, in basis points (bp), measured as the difference between the yield on 10 years maturity corporate bonds and the rate of return on treasuries. QML is the average percentage of total debt over total debt plus the product of market price of share and number of outstanding shares. DR is the average annual default rates obtained as \(1 - \exp\left(\frac{1}{10} \cdot \log(1 - DR_{10})\right)\), where \(DR_{10}\) is the Moody’s 10 year default rate (Average issuer-weighted corporate percentage default rates by whole letter rating, 1983-2005. Source: Moody’s Investor Service (2006)). In particular, the 10 year default rates are: 0.208 for AAA, 0.415 for AA, 1.248 for A, 4.721 for BBB, 21.038 for BB, 46.931 for B, and 78.673 for C.
Table II: **Ordered Probit Model for the S&P’s credit risk classes.** Results of the ordered probit analysis of firm classification criterion. Data are from Compustat North America Industrial Annual files. Financial, insurance and real state firms (SIC code 6000-6999) and regulated utility firms (SIC code 4900-4999) are excluded. The dependent variable is the S&P’s seven (whole letter) risk classes discussed previously. It takes values from 1 (class AAA) to 7 (class C). QML (*quasi-market leverage*) is given by total debt over total debt plus the product of market price of share and number of outstanding shares. ROA (*return on assets*) is given by EBIT over total debt plus the book value of equity. *t* is the *t*-ratio. *p*-value is the probability of getting a value of the test statistic as or more extreme than that observed by chance alone, if the null hypothesis $H_0$, is true.

<table>
<thead>
<tr>
<th>Ind. Var.</th>
<th>coefficient</th>
<th>std. error</th>
<th>t</th>
<th>p-value</th>
<th>95% Conf. Interval</th>
</tr>
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<tbody>
<tr>
<td>QML</td>
<td>0.02433</td>
<td>0.0005</td>
<td>45.27</td>
<td>0.000</td>
<td>[.0233, .0254]</td>
</tr>
<tr>
<td>ROA</td>
<td>-0.2439</td>
<td>0.0283</td>
<td>-8.61</td>
<td>0.000</td>
<td>[-.2994, -.1884]</td>
</tr>
</tbody>
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Table III: **Base case parameter values for the model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
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</tr>
<tr>
<td>$\sigma$</td>
<td>0.20</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>1</td>
</tr>
<tr>
<td>$r$</td>
<td>5%</td>
</tr>
<tr>
<td>$\tau_e$</td>
<td>15%</td>
</tr>
<tr>
<td>$\tau_b$</td>
<td>25%</td>
</tr>
<tr>
<td>$\tau_c^+$</td>
<td>45%</td>
</tr>
<tr>
<td>$\tau_c^-$</td>
<td>5%</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.47</td>
</tr>
<tr>
<td>$d$</td>
<td>0.1</td>
</tr>
<tr>
<td>$F$</td>
<td>1.33</td>
</tr>
<tr>
<td>$s$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\ell$</td>
<td>0.75</td>
</tr>
<tr>
<td>$c$</td>
<td>0.60</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>0.08</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.03</td>
</tr>
<tr>
<td>$q_0$</td>
<td>0.03</td>
</tr>
<tr>
<td>$q_1$</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Figure 1: **Credit spread vs productivity.** The figure presents the section of the credit spread function, \( rb/D(k, b, \theta) - r \), with respect to the productivity parameter, \( \theta \), which is the exogenous state variable. \( D(k, b, \theta) \) is the ex coupon price of debt from Equation (13). The credit spread (in basis points/year) is determined for a given face value of debt \( b = 1.4 \) and at \( k = 7.7 \), for the three specifications of the model: GAP, FHZ and Leland. For the GAP model, we plot also two restrictions: one where disinvestment is not possible (\( k' \geq k(1 - d) \)) and one with reversible investment (\( \ell = 1 \)). The plots are obtained using \( S = 21, N_b = 35 \) and \( N_k = 71 \) points for the discretized solution.
Table IV: Investment ratio vs current debt. In this table we compute the investment ratio $k^*/k$, averaged with respect to current $k$, where $k^*$ is the optimal investment from Equation (10), for different levels of current debt face, $b$, at three possible states for firm’s productivity: low state, $\theta = 0.53$; average state, $\theta = 1$; high state, $\theta = 1.89$. We compare this ratio to the corresponding ratio for an unlevered firm, which is obtained from the GAP model with the additional constraint that it uses only equity financing.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>unlevered</th>
<th>levered</th>
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<tbody>
<tr>
<td></td>
<td>$b = 0$</td>
<td>0.88</td>
</tr>
<tr>
<td>0.53</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>1</td>
<td>1.28</td>
<td>1.33</td>
</tr>
<tr>
<td>1.89</td>
<td>1.41</td>
<td>1.67</td>
</tr>
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</table>

Table V: Volatility of the Cash Flow Rate. We compute the volatility of the cash flow rate, $\sigma_{CF}$, for three different capital levels, $k = 3$, $k = 7$, and $k = 11$, assuming that the productivity is at its long-term average value, $\theta = 1$ and the current face value of debt is $b = 3$. The cash flow for a given $k$ and $\theta$, is $CF(k) = \pi(k, \theta) - g(\pi(k, \theta) - dk - rb) - rb$, where the EBITDA, $\pi(k, \theta)$, is defined in Equation (2) and the corporate tax function, $g$, is from (3). The growth rate of cash flow is defined as $CF(k, \theta')/CF(k, 1) - 1$. $\mu_{CF}$ is the expected value of the growth rate and $\sigma_{CF}$ is the standard deviation of the cash flow rate.

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>7</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CF(k)$</td>
<td>0.2092</td>
<td>0.8809</td>
<td>1.3914</td>
</tr>
<tr>
<td>$\mu_{CF}$</td>
<td>-0.0790</td>
<td>0.0037</td>
<td>0.0108</td>
</tr>
<tr>
<td>$\sigma_{CF}$</td>
<td>1.2321</td>
<td>0.3463</td>
<td>0.2612</td>
</tr>
<tr>
<td></td>
<td>GAP</td>
<td>FHZ</td>
<td>Leland</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>------</td>
<td>--------</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.80</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.20</td>
<td>0.25</td>
<td>0.20</td>
</tr>
<tr>
<td>$r$</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>$\tau_{e}$</td>
<td>15%</td>
<td>15%</td>
<td>15%</td>
</tr>
<tr>
<td>$\tau_{b}$</td>
<td>25%</td>
<td>25%</td>
<td>25%</td>
</tr>
<tr>
<td>$\tau_{c}^+$</td>
<td>49%</td>
<td>49%</td>
<td>49%</td>
</tr>
<tr>
<td>$\tau_{c}^-$</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>$K_0$</td>
<td>-</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.48</td>
<td>0.48</td>
<td>0.47</td>
</tr>
<tr>
<td>$d$</td>
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<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$F$</td>
<td>1.34</td>
<td>1.34</td>
<td>1.36</td>
</tr>
<tr>
<td>$s$</td>
<td>0.45</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\ell$</td>
<td>0.65</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$c$</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>0.12</td>
<td>0.12</td>
<td>0.08</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.08</td>
<td>0.08</td>
<td>0.03</td>
</tr>
<tr>
<td>$q_0$</td>
<td>0.04</td>
<td>0.04</td>
<td>-</td>
</tr>
<tr>
<td>$q_1$</td>
<td>0.01</td>
<td>0.02</td>
<td>-</td>
</tr>
</tbody>
</table>

Table VI: **Best fit parameters for simulations.** These parameters are obtained starting from the base case parameters in Table III and fitting the credit spreads obtained using the procedure described in Section II.C. For the Leland model, the initial face value of debt, $B_0$, is uniformly distributed in the interval $[0,6]$. 
Figure 2: Simulation. The figure presents the patterns of productivity (the state variable) times 10, book value of asset, face value and market value of debt, equity value, for a specific scenario of the state variable using the optimal solution of the three models: Leland (top), FHZ (middle) and GAP (bottom). The solution of the optimization problem is obtained with $S = 21$, $N_b = 35$ and $N_k = 71$. The initial debt level is $b = 1.4$ for all models. $k$ is held constant at 7.7 for Leland model and for FHZ model.
### Table VII: Credit spread per rating class

Estimate of credit spreads (and the related standard error) in basis points, per risk class, based on a simulated sample of 20,000 firms for 130 years, following the procedure described in Section II.B. We drop the first 30 years to reduce the influence of the initial conditions. We used the optimal policies of the three specifications of the model (Leland, FHZ, and GAP). The optimal solution is obtained using the algorithm presented in Appendix A, with $S = 21$, $N_b = 35$ and $N_k = 71$. At every step $t$ from 30 to 129 we classify firms into seven whole letter credit classes from AAA to C by minimizing the distance between the simulated quasi-market leverage of the firm and the empirical quasi-market leverage for the specific class. Firms in default and firms which are optimally unlevered are not considered. Next, we compute the credit spread for the $\omega$-th firm at $t$ as $rb_t/D(\omega,t) - r$, where $D(\omega,t)$ is the ex coupon price and $b_t$ is the current par value of debt. Finally, we take the average of the simulated credit spreads within each class, and compute the time average of this for years from year 30 to 129. We repeat the procedure for each specification of the model. The column ‘Empirical’ is reported from Table I for convenience. The figure plots the average credit spreads for risky class for the three models and compare it to the empirical ones.

<table>
<thead>
<tr>
<th>Class</th>
<th>Empirical Mean</th>
<th>Empirical s.e.</th>
<th>GAP Mean</th>
<th>GAP s.e.</th>
<th>FHZ Mean</th>
<th>FHZ s.e.</th>
<th>Leland Mean</th>
<th>Leland s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>55</td>
<td>46</td>
<td>11</td>
<td>0.24</td>
<td>60</td>
<td>0.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AA</td>
<td>65</td>
<td>130</td>
<td>79</td>
<td>0.47</td>
<td>116</td>
<td>0.41</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>92</td>
<td>176</td>
<td>137</td>
<td>0.38</td>
<td>181</td>
<td>0.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BBB</td>
<td>149</td>
<td>239</td>
<td>219</td>
<td>0.19</td>
<td>295</td>
<td>0.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BB</td>
<td>234</td>
<td>353</td>
<td>238</td>
<td>0.72</td>
<td>369</td>
<td>0.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>675</td>
<td>542</td>
<td>273</td>
<td>0.33</td>
<td>449</td>
<td>0.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1500</td>
<td>1102</td>
<td>409</td>
<td>0.53</td>
<td>772</td>
<td>2.64</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 3: **Rating class distribution.** Distribution of risk classes from a simulated sample of 20,000 firms for 130 years, following the procedure described in Section II.B. We drop the first 30 years to reduce the influence of the initial conditions. We used the optimal policies of the FHZ and GAP specifications of the model. The optimal solution is found using the algorithm presented in Appendix A, with $S = 21$, $N_b = 35$ and $N_k = 71$. At every step $t$ from 30 to 129 we classify firms into seven whole letter credit classes from AAA to C by minimizing the distance between the simulated quasi-market leverage of the firm and the empirical quasi-market leverage for the specific class. Firms in default and firms which are optimally unlevered are not considered. We compute the relative frequency for each class. Finally, we take the time average of the simulated frequencies for each class from year 30 to 129. We repeat the procedure for each specification of the model. The ‘Empirical’ relative frequencies are from Table I.
Table VIII: Quasi-market leverage per rating class. Estimate of credit spreads (and the related standard error) in basis points, per risk class, based on a simulated sample of 20,000 firms for 130 years, following the procedure described in Section II.B. We drop the first 30 years to reduce the influence of the initial conditions. We used the optimal policies of the three specifications of the model (Leland, FHZ, and GAP). The optimal solution is obtained using the algorithm presented in Appendix A, with $S = 21$, $N_b = 35$ and $N_k = 71$. At every step $t$ from 30 to 129 we classify firms into seven whole letter credit classes from AAA to C by minimizing the distance between the simulated quasi-market leverage of each firm and the empirical quasi-market leverage for the specific class. Firms in default and firms which are optimally unlevered are not considered. We compute the quasi-market leverage for the $\omega$-th firm at $t$ as $b_t/(b_t + E(\omega,t))$, where $E(\omega,t)$ is the price of equity and $b_t$ is the current par value of debt. Finally, we take the average of the simulated credit spreads within each class, and compute the time average of this for years from year 30 to 129. We repeat the procedure for each specification of the model. The column ‘Empirical’ is reported from Table I for convenience. To ease intuition, the figure plots the average quasi-market leverage for risky class for the three models and compare it to the empirical ones.
<table>
<thead>
<tr>
<th>Class</th>
<th>Empirical</th>
<th>GAP</th>
<th>FHZ</th>
<th>Leland</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02</td>
<td>0.01</td>
<td>0.26</td>
<td>0.07</td>
</tr>
<tr>
<td>AA</td>
<td>0.04</td>
<td>0.22</td>
<td>0.25</td>
<td>0.83</td>
</tr>
<tr>
<td>A</td>
<td>0.13</td>
<td>0.24</td>
<td>1.08</td>
<td>1.08</td>
</tr>
<tr>
<td>BBB</td>
<td>0.48</td>
<td>0.31</td>
<td>1.95</td>
<td>1.15</td>
</tr>
<tr>
<td>BB</td>
<td>2.33</td>
<td>0.44</td>
<td>2.63</td>
<td>1.18</td>
</tr>
<tr>
<td>B</td>
<td>6.14</td>
<td>3.22</td>
<td>7.25</td>
<td>1.33</td>
</tr>
<tr>
<td>C</td>
<td>14.32</td>
<td>28.51</td>
<td>13.52</td>
<td>15.15</td>
</tr>
</tbody>
</table>

Table IX: Default rates per rating class. Estimates of the per-year average default rate per whole letter risk class from a simulated sample of 20,000 firms for 130 years, following the procedure described in Section II.B. We drop the first 30 years to reduce the influence of the initial conditions. We used the optimal policies of the FHZ and GAP specifications of the model. The optimal solution is obtained using the algorithm presented in Appendix A, with $S = 21$, $N_b = 35$ and $N_k = 71$. At every step $t$ from 30 to 129 we classify firms into seven whole letter credit classes from AAA to C by minimizing the distance between the simulated quasi-market leverage of the firm and the empirical quasi-market leverage for the specific class. Firms in default and firms which are optimally unlevered are not considered. We compute the annual default rate for each class as the default relative frequency on the number of non-default firms at the beginning of the year. Finally, we take the time average of the simulated frequencies for each class from year 30 to 129. We repeat the procedure for each specification of the model. The ‘Empirical’ average annual default rate is reported from Table I for convenience. To help intuition, the figure plots the default rate for risky class for the three models and compare it to the empirical ones.